Introduction to Real Analysis

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Introduction to Real Analysis

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Notes for the Honors freshman calculus class at the University of Notre Dame.
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Contents

Introduction

These are notes for the Freshman Honors Calculus course at the University of Notre Dame. The word "calculus" is a misnomer since this course was intended to be an introduction to real analysis or, if you like, "calculus with proofs".

For most students this class is the first encounter with mathematical rigor and it can be a bit disconcerting. In my view the best way to overcome this is to confront rigor head on and adopt it as standard operating procedure early on. This makes for a bumpy early going, but with a rewarding payoff.

A proof is an argument that uses the basic rules of Aristotelian logic and relies on facts everyone agrees to be true. The course is based on these basic rules of the mathematical discourse. It starts from a meagre collection of obvious facts (postulates) and ends up constructing the main contours of the impressive edifice called real analysis.

No prior knowledge of calculus is assumed, but being comfortable performing algebraic manipulations is something that will make this journey more rewarding.

In writing these notes I have benefitted immensely from the students who took the Honors Calc Course during the academic years 2013-2014, 2014-2015. Their questions and reactions in class, and their expert editing have improved the original product. I asked a lot of them and I got a lot in return. I want to thank them for their hard work, curiosity and enthusiasm which made my job so much more enjoyable.

This is probably not the final form of the notes, but close to final. I will probably adjust them here and there, taking into account the feedback from future students.

Notre Dame, October 15, 2015.

The Greek Alphabet

A	α	Alpha	N	ν	Nu
B	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	O	o	Omicron
Δ	δ	Delta	Π	π	Pi
E	ε	Epsilon	P	ρ	Rho
Z	ζ	Zeta	Σ	σ	Sigma
H	η	Eta	T	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
I	ι	Iota	Φ	φ	Phi
K	κ	Kappa	X	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
M	μ	Mu	Ω	ω	Omega

Chapter 1

The basics of mathematical reasoning

1.1. Statements and predicates

Mathematics deals in *statements*. These are sentences that have a definite truth value. What does this mean? The classical text [8] does a marvelous job explaining this point of view. I will not even attempt a rigorous or exhaustive explanation. Instead, I will try to suggest it to you through examples.

Example 1.1. (a) Consider the following sentence: "*if you walk in the rain without an umbrella, you will get wet*". This is a true sentence and we say that its truth value is $TRUE$ or T. This is an example of a *statement*.

(b) Consider the sentence: *"the number x is bigger than the number y"* or, in mathematical notation, $x > y$. This sentence could be $TRUE$ or $FALSE$ (F), depending on the choice of x and y. This is not a statement because it does not have a definitive truth value. It is a type of sentence called *predicate* that is encountered often in mathematics.

A *predicate* or *formula* is a sentence that depends on some parameters (or variables). In the above example the parameters were x and y. For some choices of parameters (or variables) it becomes a $TRUE$ statement, while for other values it could be $FALSE$.

When expressed in everyday language, statements and predicates must contain a verb.

Often a predicate comes in the guise of a *property*. For example the property "*the integer n is even*" stands for the predicate "*the integer n is twice the integer m*".

(c) Consider the following sentence: *"This sentence is false."* Is this sentence true? Clearly it cannot be true because if it were, then we would conclude that the sentence is false. Thus the sentence is false so the opposite must be true, i.e., the sentence is true. Something is obviously amiss. This type of sentence is *not* a statement because it does not have

a truth value, and it is also *not* a predicate. It is a *paradox*. Paradoxes are to be avoided in mathematics. □

✍ **Notation.** It is time to explain the usage of the notation :=. For example an expression such as

$$x := \text{bla-bla-bla}$$

reads "*x is defined to be bla-bla-bla*", or "*x is short-hand for bla-bla-bla*".

The manipulations of statements and predicates are governed by the rules of *Aristotelian logic*. This and the following section will provide you with a very sparse introduction to logic. For more details and examples I refer to [**13**].

All the predicates used in mathematics are obtained from simpler ones called *atomic predicates* using the following *logical operators*.

- $NEGATION$ $\neg$ (read as *not*).
- $CONJUNCTION$ $\wedge$ (read as *and*).
- $DISJUNCTION$ $\vee$ (read as *or*).
- $IMPLICATION$ $\Rightarrow$ (read as *implies*).

To describe the effect of these operations we need to look Table 1.1 describing the *truth tables* of these operations.

p	T	F
$\neg p$	F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.1. The truth tables of $\neg, \wedge, \vee, \Rightarrow$

Here is is how one reads Table 1.1. When p is true (T), then $\neg p$ must be false (F), and when p is false, then $\neg p$ is true. To put it in simpler terms

$$\neg T = F, \ \ \neg F = T.$$

The truth table for $\wedge$ can be presented in the simplified form

$$T \wedge T = T, \ \ T \wedge F = F \wedge T = F \wedge F = F.$$

Observe that the predicate $p \vee q$ is true when at least one of the predicates p and q is true. *It is NOT an exclusive OR.* Another way of saying this is

$$T \vee T = T \vee F = F \vee T = T, \ \ F \vee F = F.$$

The equivalence $\Leftrightarrow$ is the operation

$$p \Leftrightarrow q := (p \Rightarrow q) \wedge (q \Rightarrow p).$$

Its truth table is described in Table 1.2

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.2. The truth table of $\Leftrightarrow$

Remark 1.2. (a) In everyday language, when we say that p *implies* q we mean that the statement $p \Rightarrow q$ is true. This signifies that either both p and q are true, or that p is false. Often we express this in the conditional form *if* p*, then* q.

If the implication $p \Rightarrow q$ is true, then we say that q *is a necessary condition for* p and that p *is a sufficient condition for* q. In everyday language the implications are the *if* bla-bla, *then* bla-bla statements.

The truth table for $\Rightarrow$ hides certain subtleties best illustrated by the following example. Consider the statement

$$s := \text{if an elephant can fly, then it can also drive a car.}$$

This statement is composed of two simpler statements

$$p := \text{an elephant can fly}, \quad q := \text{an elephant can drive a car.}$$

We note that the statement s coincides with the implication $p \Rightarrow q$. Obviously, both statements p and q are false, but according to the truth table for $\Rightarrow$, the implication $p \Rightarrow q$ is true, and thus s is true as well. This conclusion is rather unsettling. It may be easier to accept it if we rephrase s as follows:

if you can show me a flying elephant, then I can show you that it can also drive a car.

(b) In everyday language when we say that p *is equivalent to* q we mean that the statement $p \Leftrightarrow q$ is true. This signifies that either both p and q are true, or both are false. If p is equivalent to q, we say that q *is a necessary and sufficient condition for* p and that p *is a necessary and sufficient condition for* q.

We often express this in one of the following forms: p *if and only if* q. The mathematicians' abbreviation for the oft encountered construct "*if and only if*" is *iff*. □

Example 1.3. Consider the following predicate.

$s :=$ *if you do not clean your room, then you will not go to the movies.*

This is composed of two simpler predicates

- $p :=$ you do not clean the room.
- $q :=$ you do not go to the movies.

Observe that s is the compound predicate $p \Rightarrow q$. For s to be true, one of the following two mutually exclusive situations must happen

- either you do not clean your room AND you do not go to the movies A
- or you clean the room.

Note that there is no implied guarantee that if you clean your room, then you go to the movies. □

Example 1.4. Consider the following *true* statement: mathematicians like to be precise.

First, let us phrase this in a less ambiguous way. The above statement can be equivalently rephrased as: if you are a mathematician, then you are precise. To put it in symbolic terms

$$\underbrace{\text{you are a mathematician}}_{p} \Rightarrow \underbrace{\text{you are precise}}_{q}.$$

Thus, to be a mathematician it is necessary to be precise and to be precise it suffices to be a mathematician. However, to be precise it is not necessary to be a mathematician. □

A *tautology* is a compound predicate which is true no matter what the truth values of its atoms are.

Example 1.5. The predicate $p \vee \neg p$ is a tautology. In other words, in mathematics, a statement is either true, or false. There is no in-between. □

Two compound predicates P and Q are called *equivalent*, and we indicate this with the notation $P \longleftrightarrow Q$, if they have identical truth tables. In other words, P and Q are equivalent if the compound predicate $P \Longleftrightarrow Q$ is a tautology.

Example 1.6. Let us observe that the compound predicate $p \Rightarrow q$ is equivalent to the compound statement $(\neg p) \vee q$, i.e.

$$p \Rightarrow q \longleftrightarrow (\neg p) \vee q. \tag{1.1}$$

Indeed if p is false then $p \Rightarrow q$ and $\neg p$ are true, no matter what q. In particular $(\neg p) \vee q$ is also true, no matter what q. If p is true, then $\neg p$ is false, and we deduce that $p \Rightarrow q$ and $(\neg p) \vee q$ are either simultaneously true, or simultaneously false. □

1.2. Quantifiers

Example 1.7. Consider the following property of a person x

$$x \textit{ is at least } 6ft \textit{ tall.}$$

This does not have a definite truth value because the truth value depends on the person x. However the claims

$$C_1 := \textit{there exists a} \text{ person } x \text{ that is at least } 6ft \text{ tall},$$

and

$$C_2 := \textit{any} \text{ person } x \text{ is at least } 6ft \text{ tall}$$

have definite truth values. The claim C_1 is true, while the claim C_2 is false. □

Example 1.8. Consider the following property involving the numbers x, y

$$x > y.$$

This does not have a definite truth value. However, we can modify it to obtain statements that have definite truth values. Here are several possible modifications. (Below and in the sequel s.t. stands for *such that*)

$$S_1 := \underline{\text{For any}}\ x,\ \underline{\text{for any}}\ y,\ x > y.$$

$$S_2 := \underline{\text{For any}}\ x,\ \underline{\text{there exists}}\ y \text{ s.t. } x > y.$$

$$S_3 := \underline{\text{There exists}}\ y \text{ s.t. } \underline{\text{for any}}\ x, \quad x > y.$$

Observe that the statements S_1 and S_3 are false, while S_2 is a true statement. Notice a *very important fact*. The statement S_3 is obtained from S_2 by a seemingly innocuous transformation: we changed the order of some words. However, in doing so, we have dramatically altered the meaning of the statement. *Let this be a warning!* □

The expressions *for any*, *for all*, *there exists*, *for some* appear very frequently in mathematical communications and for this reason they were given a name, and special abbreviations. These expressions are called *quantifiers* and they are abbreviated as follows.

$$\forall := \text{for any, for all,}$$

$$\exists := \text{there exists, there exist, for some.}$$

The symbol $\forall$ is also called *the universal quantifier*, while the symbol $\exists$ is called *the existential quantifier*. There is another quantifier encountered quite frequently namely

$$\exists! := \text{there exists a unique.}$$

The above examples illustrate the roles of the quantifiers: they are used to convert predicates, which have no definite truth value, to statements which have definite truth value. To achieve this, we need to attach a quantifier to each variable in the predicate. In Example 1.8 we used a quantifier for the variable x and a quantifier for the variable y. We cannot overemphasize the following fact.

☞ ***The meaning of a statement is sensitive to the order of the quantifiers in that statement!***

Example 1.9. Let us put to work the above simple principles in a concrete situation. Consider the statement:

S := *there is a person in this class such that, if he or she gets an A in the final, then everyone will get an A in the final.*

Is this a true statement or a false statement? There are two ways to decide this. The fastest way is to think of the persons who get the lowest grade in the final. If those persons get A's, then, obviously, everybody else will get A's.

We can use a more formal way of deciding the truth value of the above statement. Consider the predicate $P(x)$:=*the person x gets an A in the final.* The quantified form of S is then

$$\exists x: \quad \big(P(x) \Rightarrow \forall y P(y) \big).$$

As we know an implication $p \Rightarrow q$ is equivalent to the disjunction $\neg p \vee q$; see (1.1). We can rewrite the above statement as

$$\exists x: \quad \big(\neg P(x) \vee \forall y P(y) \big).$$

In everyday language the above statement says that either there is a person who did not get an A or everybody gets an A. This is a *Duh!* statement or, as mathematicians like to call it, a *tautology*. □

Let us discuss how to concretely describe the negation of a statement involving quantifiers. Take for example the statements S_1, S_2, S_3 in Example 1.8. Their opposites are

$$\neg S_1 := \underline{\text{There exists}}\ x,\ \underline{\text{there exists}}\ y \text{ s.t. } x \leq y,$$

$$\neg S_2 := \underline{\text{There exists}}\ x \text{ s.t. } \underline{\text{for any}}\ y\text{: } x \leq y$$

,

$$\neg S_3 := \underline{\text{For any }}\ y,\ \underline{\text{there exists}}\ x \text{ s.t. } x \leq y.$$

Observe that all the opposites were obtained by using the following simple operations.

- Globally replacing the existential quantifier $\exists$ with its opposite $\forall$.
- Globally replace the universal quantifier $\forall$ with its opposite, $\exists$.
- Replace the predicate $x > y$ with its opposite, $x \leq y$.

When dealing with more complex statements it is very useful to remember the above rules. We summarize them below.

☞ ***The opposite of a statement that contains quantifiers is obtained by replacing each quantifier with its opposite, and each predicate with its opposite.***

Example 1.10. Consider the following portion of a famous Abraham Lincoln quote: *you can fool all of the people some of the time.* There are two conceivable ways of phrasing this rigorously.

1. *For any person y there exists a moment of time t when y can be fooled by you at time t.*

2. *There exists a moment of time t such that any person y there can be fooled by you at time t.*

We can now easily transform these into *quantified statements*.

1. $S_1 := \forall$ *person y, $\exists$ moment t, s.t., y can be fooled by you at time t.*

2. $S_2 := \exists$ *moment t, s.t, $\forall$ person y, y can be fooled by you at time t.*

Note that the two statements carry different meanings. Which do you think was meant by Lincoln? Observe also

$\neg S_1 := \exists$ *person y s.t. $\forall$ moment t: y cannot be fooled by you at time t.*

In plain English this reads: *some people cannot be fooled at any time.* □

1.3. Sets

Now that we learned a bit about the language of mathematics, let us mention a few fundamental concepts that appear in all the mathematical discourses. The most important concept is that of *set*.

Any attempt at a rigorous definition of the concept of set unavoidably leads to treacherous logical and philosophical marshes.[1] A more productive approach is not to define what a set is, but agree on a list of "uncontroversial" properties (or *axioms*) our intuition tells us the sets ought to satisfy. [2] Once these axioms are adopted, then the entire edifice of mathematics should be built on them. I refer to [**9**] for a detailed description of this point of view.

The axiomatic approach mentioned above is very labour intensive, and would send us far astray. Our goal for now is a bit more modest. We will adopt a more elementary (or naive) approach relying on the intuition of a set X as a collection of objects, usually referred to as the *elements of* X. In mathematics, a set is described by the "list" of its elements enclosed by brackets. In this list, no two objects are identical. For example, the set

$$\{winter, spring, summer, fall\}$$

is the set of seasons in a temperate region such as in Indiana. However, the list

$$\{winter, winter, spring\},$$

is not a set.

We will use the notation $x \in X$ (or $X \ni x$) to indicate that the *object* x *belongs to* the *set* X, i.e., the object x is an element of X. The notation $x \notin X$ indicates that x is not an element of X. Two sets A and B are considered identical if they consist of the same elements, i.e., the following (quantified) statement is true

$$\forall x \; \big(\, x \in A \Longleftrightarrow x \in B \,\big).$$

In words, an object belongs to A iff it also belongs to B. For example, we have the equality of sets

$$\{winter, spring, summer, fall\} = \{spring, summer, fall, winter\}.$$

There exists a distinguished set, called the *empty set* and denoted by $\emptyset$. Intuitively, $\emptyset$ is the set with no elements.

Remark 1.11. The nature of the elements of a set is not important in set theory. In fact, the elements of a set can have varied natures. For example we have the set

$$\big\{\, 1,\; \emptyset,\; \textit{apple} \,\big\}$$

which consists of three elements: of the number 1, the empty set, and the word *apple*. Another more subtle example is the set $\{\emptyset\}$ which consists of the single element, the empty set $\emptyset$. Let us observe that $\emptyset \neq \{\emptyset\}$. □

We say that a set A is a *subset* of B, and we write this $A \subset B$, if any element of A is also an element of B. In other words, $A \subset B$ signifies that the following statement is true

$$\forall x \; \big(\, x \in A \Rightarrow x \in B \,\big).$$

A *proper subset* of B is a subset $A \subset B$ such that $A \neq B$. We will use the notation $A \subsetneq B$ to indicate that A is a proper subset of B.

The *union* of two sets A, B is a new set denoted by $A \cup B$. More precisely,

$$x \in A \cup B \iff (x \in A) \vee (x \in B).$$

[1] For more details on the possible traps; see Wikipedia's article on set theory.

[2] See the above footnote.

The *intersection* of two sets A, B is a new set denoted by $A \cap B$. More precisely,

$$x \in A \cap B \iff (x \in A) \wedge (x \in B).$$

The sets A and B are said to be *disjoint* if $A \cap B = \emptyset$.

More generally, if $(A_i)_{i \in I}$ is a collection of sets, then we can define their union

$$\bigcup_{i \in I} A_i := \{\, x;\ \ \exists i \in I:\ \ x \in A_i \,\},$$

and their intersection

$$\bigcap_{i \in I} A_i := \{\, x;\ \ \forall i \in I;\ \ x \in A_i \,\}.$$

The *difference* between a set A and a set B is a new set $A \setminus B$ defined by

$$x \in A \setminus B \iff (x \in A) \wedge (x \notin B).$$

If A is a subset of X, then we will use the alternative notation $C_X A$ when referring to the difference $X \setminus A$. The set $C_X A$ is called the *complement of* A *in* X. Observe that

$$C_X\big(\, C_X A\,\big) = A.$$

It is sometimes convenient to visualize sets using *Venn diagrams*. A Venn diagram identifies a set with a region in the plane.

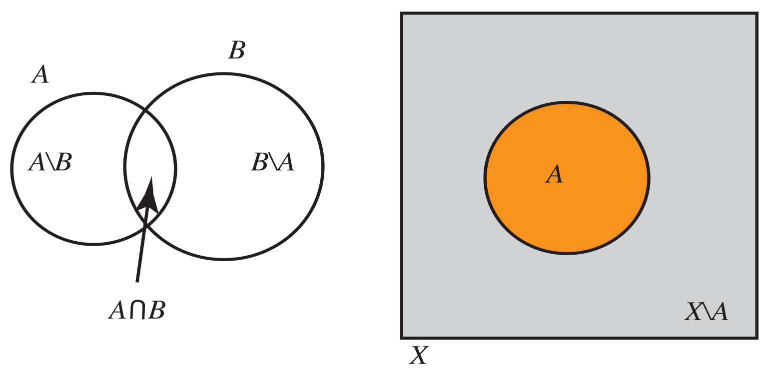

Figure 1.1. *Venn diagrams.*

Proposition 1.12 (De Morgan Laws). *If A, B are subsets of a set X then*

$$C_X(A \cup B) = (C_X A) \cap (C_X B),\ \ C_X(A \cap B) = (C_X A) \cup (C_X B). \qquad \square$$

Given two sets A and B we can form a new set $A \times B$ which consists of all ordered pairs of objects (a, b) where $a \in A$ and $b \in B$. The set $A \times B$ is called the *Cartesian product* of A and B.

Remark 1.13. As a curiosity, and to give you a sense of the intricacies of the axiomatic set theory, let us point out that above the concept of *ordered pair*, while intuitively clear, it is not rigorous. One rigorous definition of an ordered pair is due to *Norbert Wiener* who defined the ordered pair (a, b) to be the set consisting of two elements that are themselves sets: one element is the set $\{a, \emptyset\}$ and the other element is the set $\{\,\{b\}\,\}$, i.e.,

$$(a, b) := \big\{\, \{a, \emptyset\}, \{\,\{b\}\,\}\,\big\}. \qquad \square$$

Most of the time sets are defined by properties. For example, the interval $[0,1]$ consists of the real numbers x satisfying the property

$$P(x) := (x \geq 0) \wedge (x \leq 1).$$

As we discussed in the previous section, a synonym for the term *property* is the term *predicate*. Proving that an object satisfying a property P also satisfies a property Q is tantamount to showing that the set of objects satisfying property P is contained in the set of objects satisfying property Q.

Remark 1.14. To prove that two sets A and B are equal one has to prove two inclusions: $A \subset B$ and $B \subset A$. In other words one has to prove two facts:

- If x is in A, then x is also in B.
- If x is in B, then x is also in A.

□

1.4. Functions

Suppose that we are given two sets X, Y. Intuitively, a *function* f from X to Y is a "device" that feeds on elements of X. Once we feed this machine an element $x \in X$ it spits out an element of Y denoted by $f(x)$. The elements of X are called *inputs*, and those of Y, *outputs*. In Figure 1.2 we have depicted such a device. Each arrow starts at some input and its head indicates the resulting output when we feed that input to the function f.

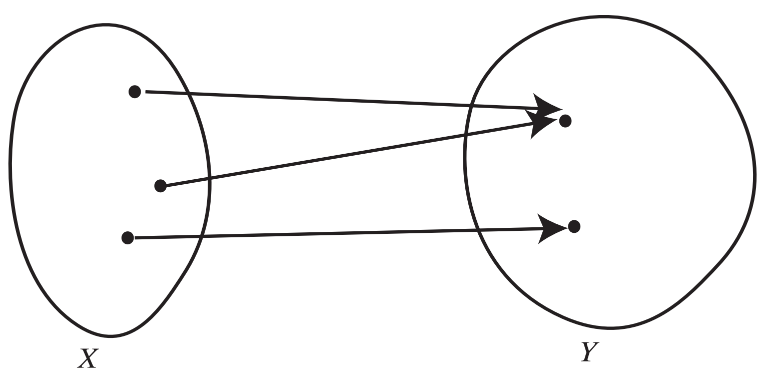

Figure 1.2. *A Venn diagram depiction of a function from X to Y.*

The above definition may not sound too academic, but at least it gives an idea of what a function is supposed to do. Mathematically, a function is described by listing its effect on each and every one of the inputs $x \in X$. The result is a list G which consists of pairs $(x,y) \in X \times Y$, where the appearance of a pair (x,y) in the list indicates the fact that when the device is fed the input x, the output will be y. Note that the list G is a subset of $X \times Y$ and has two properties.

- For any $x \in X$ there exists $y \in Y$ such that $(x,y) \in G$. Symbolically

$$\forall x \in X \;\; \exists y \in Y : \;\; (x,y) \in G. \tag{F_1}$$

- For any $x \in X$ and any $y_1, y_2 \in Y$, if $(x, y_1), (x, y_2) \in G$, then $y_1 = y_2$. Symbolically

$$\forall x \in X,\ \ \forall y_1, y_2 \in Y, \Big(\, (x, y_1) \in G \wedge (x, y_2) \in G \,\Big) \Rightarrow (y_1 = y_2). \tag{F_2}$$

Property F_1 states that to any input there corresponds at least one output, while property F_2 states that each input has at most one output.

We can use any symbol to name functions. The notation $f : X \to Y$ indicates that f is a function from X to Y. Often we will use the alternate notation $X \xrightarrow{f} Y$ to indicate that f is a function from X to Y. In mathematics there are many synonyms for the term function. They are also called *maps*, *mappings*, *operators*, or *transformations*.

Given a function $f : X \to Y$ we will refer to the set of inputs X as the *domain* of the function. The set Y is called the *codomain* of f. The result of feeding f the input $x \in X$ is denoted by $f(x)$. By definition $f(x) \in Y$. We say that x *is mapped to* $f(x)$ by f. The set

$$G_f := \big\{\, (x, f(x));\ \ x \in X \,\big\} \subset X \times Y$$

lists the effect of f on each possible input $x \in X$, and it is usually referred to as the *graph* of f.

The *range* or *image* of a function $f : X \to Y$ is the set of all outputs of f. More precisely, it is the subset $f(X)$ of F defined by

$$f(X) := \big\{ y \in Y;\ \ \exists x \in X :\ \ y = f(x) \,\big\}.$$

The range of f is also denoted by $\boldsymbol{R}(f)$. More generally, for any subset $A \subset X$ we define

$$f(A) = \big\{\, y \in Y;\ \ \exists a \in A;\ \ f(a) = y \,\big\} \subset Y. \tag{1.2}$$

The set $f(A)$ is called the *image* of A via f.

For a subset $S \subset Y$, we define the *preimage* of S via f to be the set of all inputs that are mapped by f to an element in S. More precisely the preimage of S is the set

$$f^{-1}(S) := \big\{ x \in X;\ \ f(x) \in S \,\big\} \subset X. \tag{1.3}$$

When S consists of a single point, $S = \{y_0\}$ we use the simpler notation $f^{-1}(y_0)$ to denote the preimage of $\{y_0\}$ via f. The set $f^{-1}(y_0)$ is a subset of X called the *fiber of* f *over* y_0.

A function $f : X \to Y$ is called *surjective*, or *onto*, if $f(X) = Y$. Using the visual description of a function given in Figure 1.2 we see that a function is onto if any element in Y is hit by an arrow originating at some element $x \in X$. Symbolically

$$f : X \to Y \text{ is surjective} \Longleftrightarrow \forall y \in Y,\ \ \exists x \in X :\ \ y = f(x).$$

A function $f : X \to Y$ is called *injective*, or *one-to-one*, if different inputs have different outputs under f. More precisely

$$f : X \to Y \text{ is injective} \Longleftrightarrow \forall x_1, x_2 \in X :\ \ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

$$\Longleftrightarrow \forall x_1, x_2 \in X :\ \ f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

A function $f : X \to Y$ is called *bijective* if it is both injective and surjective. We see that

$$f : X \to Y \text{ is bijective} \Longleftrightarrow \forall y \in Y\ \ \exists!\, x \in X :\ \ y = f(x).$$

Example 1.15. (a) For any set X we denote by $\mathbb{1}_X$ or by e_X the function $X \to X$ which maps any $x \in X$ to itself. The function $\mathbb{1}_X$ is called the *identity map*. The identity map is clearly injective.

(b) Suppose that X, Y are two sets. We denote π_X the mapping $X \times Y \to X$ which sends a pair (x, y) to x. We say that π_X is the *natural projection* of $X \times Y$ onto X.

(c) Given a function $f : X \to Y$ and a subset $A \subset X$ we can construct a new function $f|_A : A \to Y$ called the *restriction* of f to A and defined in the obvious way

$$f|_A(a) = f(a), \ \ \forall a \in A.$$

(d) If X is a set and $A \subset X$, then we denote by i_A the function $A \to X$ defined as the restriction of $\mathbb{1}_X$ to A. More precisely

$$i_A(a) = a, \ \ \forall a \in A.$$

The function i_A is called the *natural inclusion map* associated to the subset $A \subset X$. □

Given two functions

$$X \xrightarrow{f} Y, \ \ Y \xrightarrow{g} Z$$

we can form their *composition* which is a function $g \circ f : X \to Z$ defined by

$$g \circ f(x) := g\big(f(x) \big).$$

Intuitively, the action of $g \circ f$ on an input x can be described by the diagram

$$x \overset{f}{\mapsto} f(x) \overset{g}{\mapsto} g\big(f(x) \big).$$

In words, this means the following: take an input $x \in X$ and drop it in the device $f : X \to Y$; out comes $f(x)$, which is an element of Y. Use the output $f(x)$ as an input for the device $g : Y \to Z$. This yields the output $g\big(f(x) \big)$.

Proposition 1.16. *Let $f : X \to Y$ be a function. The following statements are equivalent.*

(i) *The function f is bijective.*

(ii) *There exists a function $g : Y \to X$ such that*

$$f \circ g = \mathbb{1}_Y, \ \ g \circ f = \mathbb{1}_X. \tag{1.4}$$

(iii) *There exists a **unique** function $g : Y \to X$ satisfying (1.4).*

Proof. (i) ⇒ (ii) Assume (i), so that f is bijective. Hence, for any $y \in Y$ there exists a unique $x \in X$ such that $f(x) = y$. This unique x depends on y and we will denote it by $g(y)$; see Figure 1.3.

The correspondence $y \mapsto g(y)$ defines a function $g : Y \to X$. By construction, if $x = g(y)$, then

$$y = f(x) = f\big(g(y) \big) = f \circ g(y) \ \ \forall y \in Y$$

so that $f \circ g = \mathbb{1}_Y$. Also, if $y = f(x)$, then

$$x = g(y) = g\big(f(x) \big) = g \circ f(x), \ \ \forall x \in X.$$

Hence $g \circ f = \mathbb{1}_X$. This proves the implication (i) ⇒ (ii)

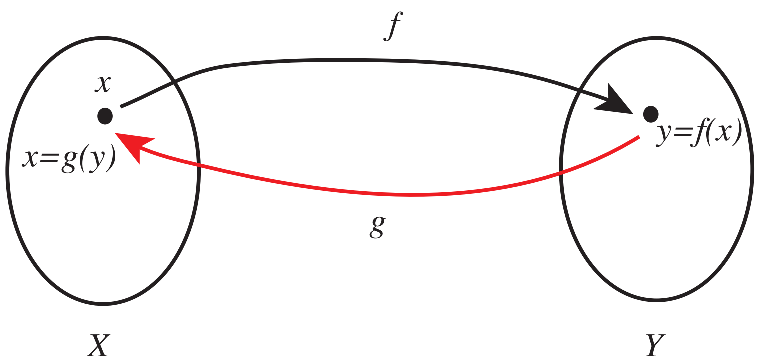

Figure 1.3. *Constructing the inverse of a bijective function $X \to Y$.*

(ii) $\Rightarrow$ (iii) Assume (i). We need to show that if $g_1, g_2 : Y \to X$ are two functions satisfying (1.4), then $g_1 = g_2$, i.e., $g_1(y) = g_2(y)$, $\forall y \in Y$.

Let $y \in Y$. Set $x_1 = g_1(y)$. Then

$$f(x_1) = f\big(\, g_1(y)\,\big) = f \circ g_1(y) \stackrel{(1.4)}{=} y.$$

On the other hand,

$$g_2(y) = g_2\big(\, f(x_1)\,\big) = g_2 \circ f(x_1) \stackrel{(1.4)}{=} x_1 = g_1(y).$$

This proves the implication (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i). We assume that there exists a function $g : Y \to X$ satisfying (1.4) and we will show that f is bijective. We first prove that f is injective, i.e.,

$$\forall x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

Indeed, if $f(x_1) = f(x_2)$, then

$$x_1 \stackrel{(1.4)}{=} g \circ f(x_1) = g\big(\, f(x_1)\,\big) = g\big(\, f(x_2)\,\big) = g \circ f(x_2) \stackrel{(1.4)}{=} x_2.$$

To prove surjectivity we need to show that for any $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Let $y \in Y$. Set $x = g(y)$. Then

$$y \stackrel{(1.4)}{=} f \circ g(y) = f\big(\, g(y)\,\big) = f(x).$$

This proves the surjectivity of f and completes the proof of Proposition 1.16. □

Definition 1.17. Let $f : X \to Y$ be a bijective function. The *inverse* of f is the unique function $g : Y \to X$ satisfying (1.4). The inverse of a bijective function f is denoted by f^{-1}. □

1.5. Exercises

Exercise 1.1. Show that

$$\neg(p \vee q) \longleftrightarrow (\neg p \wedge \neg q), \quad \neg(p \wedge q) \longleftrightarrow \neg p \vee \neg q. \qquad \square$$

Exercise 1.2. (a) Show that

$$(p \Rightarrow q) \longleftrightarrow (\neg q \Rightarrow \neg p), \quad \neg(p \Rightarrow q) \longleftrightarrow (p \wedge \neg q).$$

(b) Consider the predicates

$$p := \text{the elephant } x \text{ can fly}, \quad q := \text{the elephant } x \text{ can drive}.$$

Let us stipulate that p is false. Show that the predicate $p \Rightarrow q$ is true by showing that its negation $\neg(p \Rightarrow q)$ is false. □

Exercise 1.3. Consider the exclusive-OR operation $\vee^*$ with truth table

p	q	$p \vee^* q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 1.3. The truth table of "$\vee^*$"

Show that

$$(p \vee^* q) \longleftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q) \longleftrightarrow (p \Longleftrightarrow \neg q) \longleftrightarrow (p \Rightarrow \neg q) \wedge (\neg p \Rightarrow q). \qquad \square$$

Exercise 1.4 (Modus ponens)**.** Show that the compound predicate

$$\big((p \Rightarrow q) \wedge p \big) \Rightarrow q$$

is a tautology. □

Exercise 1.5 (Modus tollens)**.** Show that the compound predicate

$$\big((p \Rightarrow q) \wedge \neg q \big) \Rightarrow \neg p$$

is a tautology. □

Exercise 1.6. Translate each of the following propositions into a *quantified statement* in standard form, write its symbolic negation, and then state its negation in words. (Use Example 1.10 as guide.)

(i) You can fool some of the people all of the time.
(ii) Everybody loves somebody sometime.
(iii) You cannot teach an old dog new tricks.
(iv) When it rains, it pours.

□

Exercise 1.7. Consider the following predicates.

$$P := \text{I will attend your party.}$$

$$Q := \text{I go to a movie.}$$

Rephrase the predicate

$$\text{I will attend your party unless I go to a movie}$$

using the predicates P, Q and the logical operators $\neg, \vee, \wedge, \Rightarrow$. □

Exercise 1.8. Give an example of three sets A, B, C satisfying the following properties

$$A \cap B \neq \emptyset, \;\; B \cap C \neq \emptyset, \;\; C \cap A \neq \emptyset, \;\;\; A \cap B \cap C = \emptyset.$$ □

Exercise 1.9. Suppose that A, B, C are three arbitrary sets. Show that

$$A \cap \big(B \cup C \big) = \big(A \cap B \big) \cup \big(A \cap C \big),$$

$$A \cup \big(B \cap C \big) = \big(A \cup B \big) \cap \big(A \cup C \big),$$

and

$$A \setminus \big(B \cup C \big) = \big(A \setminus B \big) \cap \big(A \setminus C \big).$$

(In the above equalities it should be understood that the operations enclosed by parentheses are to be performed first.)

Hint. Use Remark 1.14. □

Exercise 1.10. Suppose that $f : X \to Y$ is a function and $A, B \subset Y$ are subsets of the codomain. Prove that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \;\; f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

Hint. Take into account (1.3) and Remark 1.14. □

Exercise 1.11. Let $f : X \to Y$ be a map between the sets X, Y. Prove that f is one-to-one *if and only if* for any subsets $A, B \subset X$ we have

$$f(A \cap B) = f(A) \cap f(B).$$ □

Exercise 1.12. Suppose A, B are sets and $f : A \to B$ is a map.[3] Define the maps

$$\varphi : A \to A \times B, \;\; \rho : A \times B \to B$$

by setting

$$\varphi(a) := \big(a, f(a)\big), \;\; \forall a \in A, \;\; \rho(a, b) := b, \;\; \forall (a, b) \in A \times B.$$

Prove that the following hold.

(i) The map φ is injective.

(ii) The map ρ is surjective.

(iii) $f = \rho \circ \varphi$.

□

[3] Recall that a map is a function.

Exercise 1.13. Suppose that $f : X \to Y$ and $g : Y \to Z$ are two bijective maps. Prove that the composition $g \circ f$ is also bijective and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

□

Exercise 1.14. Suppose that $f : X \to Y$ is a function. Prove that the following statements are equivalent.

(i) The function f is injective.

(ii) There exists a function $g : Y \to X$ such that $g \circ f = \mathbb{1}_X$.

Exercise 1.15. Suppose that $f : X \to Y$ is a function. Prove that the following statements are equivalent.

(i) The function f is surjective.

(ii) There exists a function $g : Y \to X$ such that $f \circ g = \mathbb{1}_Y$.

□

Chapter 2

The Real Number System

Any attempt to define the concept of number is fraught with perils of a logical kind: we will eventually end up chasing our tails. Instead of trying to explain *what numbers are* it is more productive to explain *what numbers do*, and *how they interact with each other*.

In this section we gather in a coherent way some of the basic properties our intuition tells us that real numbers ought to satisfy. We will formulate them precisely and we will declare, by fiat, that *these are true statements*. We will refer to these as the *axioms of the real number system*. (Things are a bit more subtle, but that's the gist of our approach.) All the other properties of the real numbers follow from these axioms. Such deductible properties are known in mathematics as *Proposition*s or *Theorem*s. The term *Theorem* is used sparingly and it is reserved to the more remarkable properties.

The process of deducing new properties from the already established ones is called a mathematical *proof*. Intuitively, a proof is a complete, precise and coherent explanation of a fact. In this course we will prove all of the calculus facts you are familiar with, and much more.

2.1. The algebraic axioms of the real numbers

The first thing that we observe is that the real numbers, whatever their nature, form a set. We will encounter this set so often in our mathematical discourse that it deserves a short name and symbol. We will denote the set of real numbers by $\mathbb{R}$.

Another thing we know from experience is that we can operate with numbers. More precisely we can add, subtract, multiply and divide real numbers. Of these four operations, the addition and multiplication are the fundamental ones. These are special instances of a more general mathematical concept, that of *binary operation*.

A binary operation on a set S is, by definition, a function $S \times S \to S$. Loosely, a binary operation is a gizmo that feeds on pairs of elements of S, processes such a pair in some

fashion, and produces a single element of S. We list the first axioms describing the set of real numbers.

Axiom 1. The set $\mathbb{R}$ of real numbers $\mathbb{R}$ is equipped with two binary operations,

- *addition*
$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \;\; (x, y) \mapsto x + y,$$
- and *multiplication*
$$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \;\; (x, y) \mapsto x \cdot y.$$

□

The operation of multiplication is sometimes denoted by the symbol $\times$.

Axiom 2. The addition is *associative*, i.e.,

$$\forall x, y, z \in \mathbb{R}; \;\; (x + y) + z = x + (y + z).$$ □

The usage of parentheses ($-$) indicates that we first perform the operation enclosed by them.

Axiom 3. The addition is *commutative*, i.e.,

$$\forall x, y \in \mathbb{R} : \;\; x + y = y + x.$$ □

Axiom 4. An *additive identity element* exists. This means that there exists at least one real number u such that

$$x + u = u + x = x, \;\; \forall x \in \mathbb{R}. \tag{2.1}$$

□

Before we proceed to our next axiom, let us observe that there exists precisely one additive identity element.

Proposition 2.1. *If $u_0, \hat{u}_0 \in \mathbb{R}$ are additive identity elements, then $u_0 = \hat{u}_0$.*

Proof. Since u_0 is an identity element, if we choose $x = \hat{u}_0$ in (2.1) we deduce that

$$\hat{u}_0 + u_0 = u_0 + \hat{u}_0 = \hat{u}_0.$$

On the other hand, $\hat{u}_0$ is also an identity element and if we let $x = u_0$ in (2.1) we conclude that

$$u_0 + \hat{u}_0 = \hat{u}_0 + u_0 = u_0.$$

Thus $u_0 = \hat{u}_0$. □

Definition 2.2. The unique additive identity element of $\mathbb{R}$ is denoted by 0. □

Axiom 5. *Additive inverses exist.* More precisely, this means that for any $x \in \mathbb{R}$ there exists at least one real number $y \in \mathbb{R}$ such that

$$x + y = y + x = 0.$$ □

We have the following result whose proof is left to you as an exercise.

Proposition 2.3. *Additive inverses are unique. This means that if x, y, y' are real numbers such that*

$$x + y = y + x = 0 = x + y' = y' + x,$$

then $y = y'$. □

Definition 2.4. The unique additive inverse of a real number x is denoted by $-x$. Thus

$$x + (-x) = (-x) + x = 0, \quad \forall x \in \mathbb{R}.$$ □

Axiom 6. The multiplication is *associative*, i.e.,

$$\forall x, y, z \in \mathbb{R}; \quad (x \cdot y) \cdot z = x \cdot (y \cdot z).$$ □

Axiom 7. The multiplication is *commutative*, i.e.,

$$\forall x, y \in \mathbb{R}: \quad x \cdot y = y \cdot x.$$ □

Axiom 8. A *multiplicative identity element* exists. This means that there exists at least one *nonzero* real number u such that

$$x \cdot u = u \cdot x = x, \quad \forall x \in \mathbb{R}. \tag{2.2}$$

□

Arguing as in the proof of Proposition 2.1 we deduce that there exists precisely one multiplicative identity element. We denote it by 1. We define

$$2 := 1 + 1, \quad x^2 := x \cdot x, \quad \forall x \in \mathbb{R}. \qquad (✎)$$

Axiom 9. *Multiplicative inverses exist.* More precisely, this means that for any $x \in \mathbb{R}$, $x \neq 0$, there exists at least one real number $y \in \mathbb{R}$ such that

$$x \cdot y = y \cdot x = 1.$$ □

Proposition 2.3 has a multiplicative counterpart that states that multiplicative inverses are unique. The multiplicative inverse of the *nonzero* real number x is denoted by x^{-1}, or $1/x$, or $\frac{1}{x}$. Also, we will frequently use the notation

$$\frac{x}{y} := x \cdot y^{-1}, \quad y \neq 0.$$

☞ ***The real number zero does not have an inverse. For this reason division by zero is an illegal and very dangerous operation. NEVER DIVIDE BY ZERO!***

Axiom 10. *Distributivity.*

$$\forall x, y, z \in \mathbb{R}: \quad x \cdot (y + z) = x \cdot y + x \cdot z.$$ □

✍ *To save energy and time we agree to replace the notation $x \cdot y$ with the simpler one, xy, whenever no confusion is possible.*

Definition 2.5. A set satisfying Axioms 1 through 10 is called a *field.* □

The above axioms have a number of ”obvious” consequences.

Proposition 2.6. (i) $\forall x \in \mathbb{R}$, $x \cdot 0 = 0$

(ii) $\forall x, y \in \mathbb{R}$, $(xy = 0) \Rightarrow (x = 0) \vee (y = 0)$.

(iii) $\forall x \in \mathbb{R}$, $-x = (-1) \cdot x$.

(iv) $\forall x \in \mathbb{R}$, $(-1) \cdot (-x) = x$.

(v) $\forall x, y \in \mathbb{R}$, $(-x) \cdot (-y) = xy$.

Proof. We will prove only part (i). The rest are left as exercises. Since 0 is the additive identity element we have $0 + 0 = 0$ and

$$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0.$$

If we add $-(x \cdot 0)$ to both sides of the equality $x \cdot 0 = x \cdot 0 + x \cdot 0$ we deduce $0 = x \cdot 0$. □

2.2. The order axiom of the real numbers

Experience tells us that we can compare two real numbers, i.e., given two real numbers we can decide which is smaller than the other. In particular, we can decide whether a number is positive or not. In more technical terms we say that we can *order* the set of real numbers. The next axiom formalizes this intuition.

Axiom 11. There exists a subset $\boldsymbol{P} \subset \mathbb{R}$ called the *subset of positive real numbers* satisfying the following two conditions.

(i) If x and y are in $\boldsymbol{P}$, then so are their sum and product, $x + y \in \boldsymbol{P}$ and $xy \in \boldsymbol{P}$.

(ii) If $x \in \mathbb{R}$, then *exactly one* of the following statements is true:

$$x \in \boldsymbol{P}, \text{ or } x = 0, \text{ or } -x \in \boldsymbol{P}.$$ □

Definition 2.7. Let $x, y \in \mathbb{R}$.

(i) We say that x is *negative* if $-x \in \boldsymbol{P}$.

(ii) We say that x is *greater than* y, and we write this $x > y$ if $x - y$ is positive. We say that x is *less than* y, written $x < y$, if y is greater than x.

(iii) We say that x is *greater than or equal to* y, and we write this $x \geq y$, if $x > y$ or $x = y$. We say that x is *less than or equal to* y, and we write this $x \leq y$, if $y \geq x$.

(iv) A real number x is called *nonnegative* if $x \geq 0$.

□

Observe that $x > 0$ signifies that $x \in \boldsymbol{P}$.

Proposition 2.8. (i) $1 > 0$, *i.e.* $1 \in \boldsymbol{P}$.

(ii) *If* $x > y$ *and* $y > z$, *then* $x > z$, $x, y, z \in \mathbb{R}$.

(iii) *If* $x > y$, *then for any* $z \in \mathbb{R}$, $x + z > y + z$.

(iv) *If* $x > y$ *and* $z > 0$, *then* $xz > yz$.

(v) *If* $x > y$ *and* $z < 0$, *then* $xz < yz$.

Proof. We will prove only (i) and (ii). The proofs of the other statements are left to you as exercises. To prove (i) we argue by contradiction. Thus we assume that $1 \notin \boldsymbol{P}$. By Axiom

8, $1 \neq 0$, so Axiom 11 implies that $-1 \in \boldsymbol{P}$ and $(-1)\cdot(-1) \in \boldsymbol{P}$. Using Proposition 2.6(v) we deduce that

$$1 = (-1)\cdot(-1) \in \boldsymbol{P}.$$

We have reached a contradiction which proves (i).

To prove (ii) observe that

$$x > y \Rightarrow x - y \in \boldsymbol{P},\ \ y > z \Rightarrow y - z \in \boldsymbol{P}$$

so that

$$x - z = (x-y) + (y-z) \in \boldsymbol{P} \Rightarrow x > z.$$

□

Definition 2.9 (Intervals)**.** Let $a, b \in \mathbb{R}$. We define the following sets.

(i) $(a,b) =]a,b[:= \{\, x \in \mathbb{R};\ \ a < x < b \,\}$.
(ii) $(a,b] =]a,b] := \{\, x \in \mathbb{R};\ \ a < x \leq b \,\}$.
(iii) $[a,b) = [a,b[:= \{\, x \in \mathbb{R};\ \ a \leq x < b \,\}$.
(iv) $[a,b] := \{\, x \in \mathbb{R};\ \ a \leq x \leq b \,\}$.
(v) $[a,\infty) = [a,\infty[:= \{\, x \in \mathbb{R};\ \ a \leq x \,\}$.
(vi) $(a,\infty) =]a,\infty[:= \{\, x \in \mathbb{R};\ \ a < x \,\}$.
(vii) $(-\infty,a) =]-\infty,a[:= \{\, x \in \mathbb{R};\ \ x < a \,\}$.
(viii) $(-\infty,a] =]-\infty,a] := \{\, x \in \mathbb{R};\ \ x \leq a \,\}$.

The above sets are generically called *intervals*. The intervals of the form $[a,b]$, $[a,\infty)$, or $(-\infty,a]$ are called *closed*, while the intervals of the form (a,b), (a,∞), or $(-\infty,a)$ are called *open*. □

I would like to emphasize that in the above definition we made no claim that any or some of the intervals are nonempty. This is indeed the case, but this fact requires a proof.

Definition 2.10. For any $x \in \mathbb{R}$ we define the *absolute value* of x to be the quantity

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

□

Proposition 2.11. (i) *Let $\varepsilon > 0$. Then $|x| < \varepsilon$ if and only if $-\varepsilon < x < \varepsilon$, i.e.,*

$$(-\varepsilon,\varepsilon) = \{\, x \in \mathbb{R};\ \ |x| < \varepsilon \,\}.$$

(ii) $x \leq |x|$, $\forall x \in \mathbb{R}$.
(iii) $|xy| = |x|\cdot|y|$, $\forall x, y \in \mathbb{R}$. *In particular,* $|-x| = |x|$
(iv) $|x+y| \leq |x| + |y|$, $\forall x, y \in \mathbb{R}$.

Proof. We prove only (i) leaving the other parts as an exercise. We have to prove two things,

$$|x| < \varepsilon \Rightarrow -\varepsilon < x < \varepsilon, \tag{2.3}$$

and

$$-\varepsilon < x < \varepsilon \Rightarrow |x| < \varepsilon. \tag{2.4}$$

To prove (2.3) let us assume that $|x| < \varepsilon$. We distinguish two cases. If $x \geq 0$, then $|x| = x$ and we conclude that $-\varepsilon < 0 \leq x < \varepsilon$. If $x < 0$, then $|x| = -x$ and thus $0 < -x = |x| < \varepsilon$. This implies $-\varepsilon < -(-x) = x < 0 < \varepsilon$.

Conversely, let us assume that $-\varepsilon < x < \varepsilon$. Multiplying this inequality by -1 we deduce that $-\varepsilon < -x < \varepsilon$. If $0 \leq x$, then $|x| = x < \varepsilon$. If $x < 0$ then $|x| = -x < \varepsilon$. □

Definition 2.12. The distance between two real numbers x, y is the nonnegative number $\operatorname{dist}(x, y)$ defined by

$$\operatorname{dist}(x, y) := |x - y|. \qquad \square$$

Very often in calculus we need to solve *inequalities*. The following examples describe some simple ways of doing this.

Example 2.13. (a) Suppose that we want to find all the real numbers x such that

$$(x-1)(x-2) > 0.$$

To solve this inequality we rely on the following simple principle: the product of two real numbers is positive if and only if both numbers are positive or both numbers are negative; see Exercise 2.8. In this case the answer is simple: the numbers $(x-1)$ and $(x-2)$ are both positive iff $x > 2$ and they are both negative iff $x < 1$. Hence

$$(x-1)(x-2) > 0 \iff x \in (-\infty, 1) \cup (2, \infty).$$

(b) Consider the more complicated problem: find all the real numbers x such that

$$(x-1)(x-2)(x-3) > 0.$$

The answer to this question is also decided by the multiplicative rule of signs, but it is convenient to organize or work in a table. In each of row we read the sign of the quantity

x	$-\infty$		1		2		3		∞
$(x-1)$	$-\infty$	$----$	0	$+++$	$+$	$+++$	$+$	$+++$	∞
$(x-2)$	$-\infty$	$----$	$-$	$---$	0	$+++$	$+$	$+++$	∞
$(x-3)$	$-\infty$	$----$	$-$	$---$	$-$	$---$	0	$+++$	∞
$(x-1)(x-2)(x-3)$	$-\infty$	$----$	0	$+++$	0	$---$	0	$+++$	∞

listed at the beginning of the row. The signs in the bottom row are obtained by multiplying the signs in the column above them. We read

$$(x-1)(x-2)(x-3) > 0 \iff x \in (1, 2) \cup (3, \infty).$$

(c) Consider the related problem: find all the real numbers x such that

$$\frac{(x-1)}{(x-2)(x-3)} \geq 0.$$

Before we proceed we need to eliminate the numbers $x = 2$ and $x = 3$ from our considerations because the denominator of the above fraction vanishes for these values of x and the ***division by* 0 *is an illegal operation***. We obtain a similar table

x	$-\infty$		1		2		3		∞
$(x-1)$	$-\infty$	$----$	0	$+++$	$+$	$+++$	$+$	$+++$	∞
$(x-2)$	$-\infty$	$----$	$-$	$---$	0	$+++$	$+$	$+++$	∞
$(x-3)$	$-\infty$	$----$	$-$	$---$	$-$	$---$	0	$+++$	∞
$\frac{(x-1)}{(x-2)(x-3)}$	$-\infty$	$----$	0	$+++$	!	$---$	!	$+++$	∞

The exclamation signs at the bottom row are warning us that for the corresponding values of x the fraction has no meaning. We read

$$\frac{(x-1)}{(x-2)(x-3)} \geq 0 \iff x \in [1,2) \cup (3,\infty). \qquad \square$$

Example 2.14. We want to discuss a question involving inequalities frequently encountered in real analysis. Consider the statement

$$P(M): \ \ \forall x \in \mathbb{R}, \ \ x > M \Rightarrow \left| \frac{x^2}{x^2+x-2} - 1 \right| < \frac{1}{10}.$$

We want to show that there exists at least one positive number M such that $P(M)$ is true, i.e., we want to prove that the statement

$$\exists M > 0 \text{ such that, } \ \forall x \in \mathbb{R}, \ \ x > M \Rightarrow \left| \frac{x^2}{x^2+x-2} - 1 \right| < \frac{1}{10}.$$

Let us observe that if $P(M)$ is true and $M' \geq M$, then $P(M')$ is also true. Thus, once we find one M such that $P(M)$ is true, then $P(M')$ is true for all $M' \in [M,\infty)$.

We are content with finding only one M such that $P(M)$ is true and the above observation shows that in our search we can assume that M is very large. This is a bit vague, so let us see how this works in our special case.

First, we need to make sure that our algebraic expression is well defined so we need to require that the denominator $x^2+x-2 = (x-1)(x+2)$ is not zero. Thus we need to assume that $x \neq 1, -2$. In particular, we will restrict our search for M to numbers larger than 1. We have

$$\left| \frac{x^2}{x^2+x-2} - 1 \right| = \left| \frac{x^2-(x^2+x-2)}{x^2+x-2} \right| = \left| \frac{-x+2}{x^2+x-2} \right| = \left| \frac{x-2}{x^2+x-2} \right|.$$

Since we are investigating the properties of the last expression for $x > M > 1$ we deduce that for $x > 2$ both quantities $x-2$ and $(x-1)(x+2)$ are positive and thus

$$\left| \frac{x-2}{x^2+x-2} \right| = \frac{x-2}{x^2+x-2}.$$

We want this fraction to be small, smaller than $\frac{1}{10}$. Note that for $x > 2$ we have

$$\frac{x-2}{x^2+x-2} \leq \frac{x-1}{x^2+x-2} = \frac{x-1}{(x-1)(x+2)} = \frac{1}{x+2},$$

and

$$x > 2 \ \wedge \ \frac{1}{x+2} < \frac{1}{10} \iff x+2 > 10 \iff x > 8 \, .$$

We deduce that if $x > 8$, then

$$\frac{1}{10} > \frac{1}{x+2} > \left| \frac{x^2}{x^2+x-2} - 1 \right|.$$

Hence $P(8)$ is true. □

2.3. The completeness axiom

Definition 2.15. Let $X \subset \mathbb{R}$ be a nonempty set of real numbers.

(i) A real number M is called an *upper bound* for X if

$$\forall x \in X: \quad x \leq M. \tag{2.5}$$

(ii) The set X is said to be *bounded above* if it admits an upper bound.

(iii) A real number m is called a *lower bound* for X if

$$\forall x \in X: \quad x \geq m. \tag{2.6}$$

(iv) The set X is said to be *bounded below* if it admits a lower bound.

(v) The set X is said to be *bounded* if it is bounded both above and below.

□

Example 2.16. (a) The interval $(-\infty, 0)$ is bounded above, but not below. The interval $(0, \infty)$ is bounded below, but not above, while the interval $(0, 1)$ is bounded. □

(b) Consider the set R consisting of positive real numbers x such that $x^2 < 2$. This set is not empty because $1^2 = 1 < 2$ so that $1 \in R$. Let us show that this set is bounded above. More precisely, we will prove that

$$x^2 < 2 \Rightarrow x \leq 2.$$

We argue by contradiction. Suppose that $x \in R$ yet $x > 2$. Then

$$x^2 - 2^2 = (x-2)(x+2) > 0.$$

Hence $x^2 > 2^2 > 2$ which shows that $x \notin R$. This contradiction proves that 2 is an upper bound for R. □

Definition 2.17. Let $X \subset \mathbb{R}$ be a nonempty set of real numbers.

(i) A *least upper bound* for X is an upper bound M with the following additional property: if M' is another upper bound of X, then $M \leq M'$.

(ii) A *greatest lower bound* for X is a lower bound m with the following additional property: if m' is another lower bound of X, then $m \geq m'$.

□

Thus, M is a least upper bound for X if

- $\forall x \in X$, $x \leq M$, and
- if $M' \in \mathbb{R}$ is such that $\forall x \in X$, $x \leq M'$, then $M \leq M'$.

Proposition 2.18. *Any nonempty set $X \subset \mathbb{R}$ admits at most one least upper bound, and at most one greatest lower bound.*

Proof. We prove only the statement concerning upper bounds. Suppose that M_1, M_2 are two least upper bounds. Since M_1 is a least upper bound, and M_2 is an upper bound we have $M_1 \leq M_2$. Similarly, since M_2 is a least upper bound we deduce $M_2 \leq M_1$. Hence $M_1 \leq M_2$ and $M_2 \leq M_1$ so that $M_1 = M_2$. □

Definition 2.19. Let $X \subset \mathbb{R}$ be a nonempty set of real numbers.

(i) The least upper bound of X, when it exists, is called the *supremum* of X and it is denoted by $\sup X$.

(ii) The greatest lower bound of X, when it exists, is called the *infimum* of X and it is denoted by $\inf X$.

□

Example 2.20. Suppose that $X = [0, 1)$. Then $\sup X = 1$ and $\inf X = 0$. Note that $\sup X$ is not an element of X. □

Proposition 2.21. *Let $X \subset \mathbb{R}$ be a nonempty set of real numbers and $M \in \mathbb{R}$. The following statements are equivalent.*

(i) $M = \sup X$.

(ii) *The number M is an upper bound for X and for any $\varepsilon > 0$ there exists $x \in X$ such that $x > M - \varepsilon$.*

Proof. (i) ⇒ (ii) Assume that M is the least upper bound of X. Then clearly M is an upper bound and we have to show that for any $\varepsilon > 0$ we can find a number $x \in X$ such that $x > M - \varepsilon$.

Because M is the least upper bound and $M - \varepsilon < M$, we deduce that $M - \varepsilon$ is *not* an upper bound for X. In other words, the opposite of (2.5) must be true, i.e., there must exist $x \in X$ such that x is not less or equal to $M - \varepsilon$.

(ii) ⇒ (i) We have to show that if M' is another upper bound then $M \leq M'$. We argue by contradiction. Suppose that $M' < M$. Then $M' = M - \varepsilon$ for some positive number ε. The assumption (ii) implies that $x > M - \varepsilon$ for some number $x \in X$ so that $M' = M - \varepsilon$ is not an upper bound. We reached a contradiction which completes the proof. □

The Completeness Axiom. *Any nonempty set of real numbers that is bounded above **admits** a least upper bound.* □

From the completeness axiom we deduce the following result whose proof is left to you as Exercise 2.22.

Proposition 2.22. *If the nonempty set $X \subset \mathbb{R}$ is bounded below, then it admits a greatest lower bound.* □

Definition 2.23. Let $X \subset \mathbb{R}$ be a nonempty subset.

(i) We say that X admits a *maximal element* if X is bounded above and $\sup X \in X$. In this case we say that $\sup X$ is the maximum of X and it is denoted by $\max X$.

(ii) We say that X admits a *minimal element* if X is bounded below and $\inf X \in X$. In this case we say that $\inf X$ is called the *minimum* of X and it is denoted by $\min X$.

□

Note that the interval $I = [0, 1)$ has no maximal element, but it has a minimal element

$$\min I = 0.$$

2.4. Visualizing the real numbers

The approach we have adopted in introducing the real numbers differs from the historical course of things. For centuries scientists did not bother to ask what are the real numbers, often relying on intuition to prove things. This lead to various contradictory conclusions which prompted mathematicians to think more carefully about the concept of number and treating the intuition more carefully.

This does not mean that the intuition stopped playing an important part in the modern mathematical thinking. On the contrary, intuition is still the first guide, but it always needs to be checked and backed by rigorous arguments.

For example, you learned to visualize the numbers as points on a line called the *real line*. We will not even attempt to explain what a line is. Instead we will rely on our physical intuition of this geometric concept. The real line is more than just a line, it is a line enriched with several attributes.

- It has a distinguished point called the *origin* which should be thought of as the real number 0.
- It is equipped with an *orientation*, i.e., a direction of running along the line visually indicated by an arrowhead at one end of the line; see Figure 2.1. Equivalently, the origin splits the line into two sides, and choosing an orientation is equivalent to declaring one side to be the *positive side* and the other side to be the *negative side*. Traditionally the above arrowhead points towards the positive side; see Figure 2.1
- There is a way of measuring the distance between two points on the line.

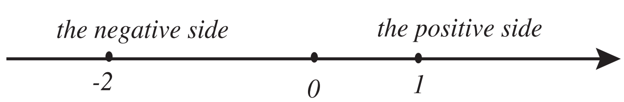

Figure 2.1. *The real line.*

For example, the number -2 can be visualized as the point on the negative side situated at distance 2 from the origin; see Figure 2.1.

Now that we have identified the set $\mathbb{R}$ of real numbers with the set of points on a line, we can visualize the Cartesian product $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ with the set of points in a plane, called the *Cartesian plane*; see the top of Figure 2.2.

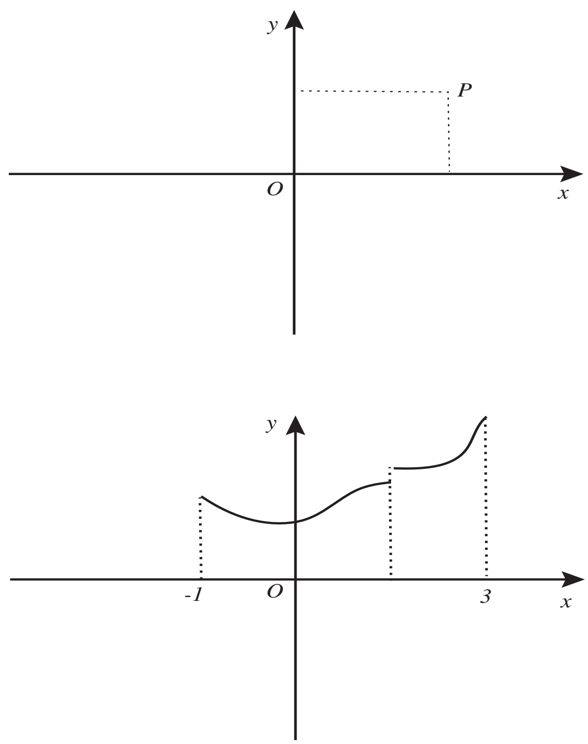

Figure 2.2. *The real line.*

Just like the real line, the Cartesian plane is more than a plane: it is a plane enriched by several attributes.

- It contains a distinguished point, called the origin and denoted by O.
- It contains two distinguished perpendicular lines intersecting at O. These lines are called the *axes* of the Cartesian plane. One of the axes is declared to be horizontal and the other is declared to be vertical.
- Each of these two axes is a real line, i.e., it has the additional attributes of a real line: each has a distinguished point, O, each has an orientation, and each is equipped with a way measuring distances along that respective line. The horizontal axis is also known as the x-axis, while the vertical one is also known as the y-axis.

The position of a point P in that plane is determined by a pair of real numbers called the *Cartesian coordinates* of that point. These two numbers are obtained by intersecting the two axes with the lines through P which are perpendicular to the axes.

An interval of the real line can be visualized as a segment on the real line, possibly with one or both endpoints removed. If I is an interval of the real line and $f : I \to \mathbb{R}$ is a function, then its graph looks typically like a curve in the Cartesian plane. For example, the bottom of Figure 2.2 depicts the graph of a function $f : [-1, 3] \to \mathbb{R}$.

2.5. Exercises

Exercise 2.1. (a) Prove Proposition 2.3.
(b) State and prove the multiplicative counterpart of Proposition 2.1. □

Exercise 2.2. Prove parts (ii)-(v) of Proposition 2.6. □

Exercise 2.3. (a) Prove that

$$(x+y)+(z+t)=\big((x+y)+z\big)+t,\ \ \forall x,y,z,t\in\mathbb{R}.$$

(b) Prove that for any $x, y, z, t, \in \mathbb{R}$ the sum $x+y+z+t$ is independent of the manner in which parentheses are inserted. □

Exercise 2.4. Prove parts (iii)-(v) of Proposition 2.8. □

Exercise 2.5. Show that for any real numbers x, y, z such that $y, z \neq 0$, we have

$$\frac{xz}{yz}=\frac{x}{y}.$$ □

Exercise 2.6. (a) Show that for any real numbers x, y, z, t such that $y, t \neq 0$ we have the equality

$$\frac{x}{y}+\frac{z}{t}=\frac{xt+yz}{yt}.$$

(b) Prove that for any real numbers x, y we have

$$x^2-y^2=(x-y)(x+y).$$

(c) Prove that the function $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^2$, is injective but not surjective. □

Exercise 2.7. Prove that if $x \leq y$ and $y \leq x$, then $x = y$. □

Exercise 2.8. (a) Prove that if $xy > 0$, then either $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$. □

(b) Prove that if $x > 0$, then $1/x > 0$.
(c) Let $x > 0$. Show that $x > 1$ if and only if $1/x < 1$.
(d) Prove that if $y > x \geq 1$, then

$$x+\frac{1}{x}<y+\frac{1}{y}.$$ □

Exercise 2.9. (a) Prove that $x^2 > 0$ for any $x \in \mathbb{R}$, $x \neq 0$.
(b) Consider the functions

$$f,g:\mathbb{R}\to\mathbb{R},\ \ f(x)=x^2+1,\ \ g(x)=2x+1.$$

Decide if any of these two functions is injective or surjective.
(c) With f and g as above, describe the functions $f \circ g$ and $g \circ f$. □

Exercise 2.10. Using the technique described in Example 2.13 find all the real numbers x such that
$$\frac{x^2}{(x-1)(x+2)} \leq 1.$$
$\square$

Exercise 2.11. (a) Find a positive number M with the following property:
$$\forall x : x > M \Rightarrow \frac{x^2}{x+1} > 10^5.$$
(b) Find a positive number M with the following property:
$$\forall x : \quad x > M \Rightarrow \frac{x^2}{x-1} > 10^6.$$
(c) Find a real number M with the following property:
$$\forall x : \quad x > M \Rightarrow \left| \frac{x^2}{(x-1)(x-2)} - 1 \right| < \frac{1}{100}.$$
$\square$

Exercise 2.12. Let $a < b$. Show that
$$a < \frac{1}{2}(a+b) < b,$$
where 2 is the real number $2 := 1 + 1$. Conclude that the interval (a, b) is nonempty. $\square$

Exercise 2.13. Prove that $x^2 + y^2 \geq 2xy$, for any $x, y \in \mathbb{R}$. Use this inequality to prove that
$$x^2 + y^2 + z^2 \geq xy + yz + zx, \quad \forall x, y, z \in \mathbb{R}.$$
$\square$

Exercise 2.14. Prove that if $0 \leq x \leq \varepsilon$, $\forall \varepsilon > 0$, then $x = 0$. (The Greek letter ε (read *epsilon*) is ubiquitous in analysis and it is almost exclusively used to denote quantities that are extremely small.) $\square$

Exercise 2.15. (a) Consider the function $f : [0, 2] \to \mathbb{R}$ given by
$$f(x) = \begin{cases} 0, & x \in [0, 1], \\ 1, & x \in (1, 2]. \end{cases}$$
Decide which of the following statements is true.

(i) $\exists L > 0$ such that $\forall x_1, x_2 \in [0, 2]$ we have $|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$.

(ii) $\forall x_1, x_2 \in [0, 2]$, $\exists L > 0$ such that $|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$.

(b) Same question, when we change the definition of f to $f(x) = x^2$, for all $x \in [0, 2]$. $\square$

Exercise 2.16. Show that for any $\delta > 0$ and any $a \in \mathbb{R}$ we have
$$(a - \delta, a + \delta) = \{\, x \in \mathbb{R}; \;\; |x - a| < \delta \,\}.$$
$\square$

Exercise 2.17. Prove the statements (ii)-(iv) of Proposition 2.11. $\square$

Exercise 2.18. Prove that for any real numbers a, b, c we have
$$\operatorname{dist}(a, c) \leq \operatorname{dist}(a, b) + \operatorname{dist}(b, c).$$
$\square$

Exercise 2.19. Prove that a set $X \subset \mathbb{R}$ is bounded if and only if there exists $C > 0$ such that $|x| \leq C$, $\forall x \in X$. □

Exercise 2.20. Fix two real numbers a, b such that $a < b$. Prove that for any $x, y \in [a, b]$ we have

$$|x - y| \leq b - a.$$ □

Exercise 2.21. State and prove the version of Proposition 2.21 involving the infimum of a bounded below set $X \subset \mathbb{R}$. □

Exercise 2.22. Let $X \subset \mathbb{R}$ be a nonempty set of real numbers. For $c \in \mathbb{R}$ define

$$cX := \left\{ cx;\ \ x \in X \right\} \subset \mathbb{R}.$$

(i) Show that if $c > 0$ and X is bounded above, then cX is bounded above and

$$\sup cX = c \sup X.$$

(ii) Show that if $c < 0$ and X is bounded above, then cX is bounded below and

$$\inf cX = c \sup X.$$

□

Exercise 2.23. (a) Let

$$A := \left\{ \frac{a}{a+1};\ \ a > 0 \right\}.$$

Compute $\inf A$ and $\sup A$.

(b) Let

$$B := \left\{ \frac{b}{b+1};\ \ b \in \mathbb{R} \setminus \{-1\} \right\}.$$

Prove that the set B is not bounded below or above. □

Chapter 3

Special classes of real numbers

3.1. The natural numbers and the induction principle

The numbers of the form

$$1, \quad 1+1, \quad (1+1)+1$$

and so forth are denoted respectively by $1, 2, 3, \dots$ and are called *natural numbers*. The term *and so forth* is rather ambiguous and its rigorous justification is provided by the *principle of mathematical induction*.

Definition 3.1. A set $X \subset \mathbb{R}$ is called *inductive* if

$$\forall x: \quad (x \in X \Rightarrow x+1 \in X).$$

$\square$

Example 3.2. The set $\mathbb{R}$ is inductive and so is any interval (a, ∞) If $(X_a)_{a \in A}$ is a collection of inductive sets, then so is their intersection

$$\bigcap_{a \in A} X_a. \qquad \square$$

Definition 3.3. The set of *natural numbers* is the smallest inductive set containing 1, i.e., the intersection of all inductive sets that contain 1. The set of natural numbers is denoted by $\mathbb{N}$. $\square$

To unravel the above definition, the set $\mathbb{N}$ is the subset of $\mathbb{R}$ uniquely characterized by the following requirements.

- The set $\mathbb{N}$ is inductive and $1 \in \mathbb{N}$.
- If $S \subset \mathbb{R}$ is an inductive set that contains 1, then $\mathbb{N} \subset S$.

The set $\mathbb{N}$ consists of the numbers

$$1,\ \ 2 := 1+1,\ \ 3 := 2+1,\ \ 4 := 3+1, \dots .$$

Note that $0 \notin \mathbb{N}$. Indeed, the interval $[1,\infty)$ is an inductive set, containing 1 and thus must contain $\mathbb{N}$. On the other hand, this interval does not contain 0. The above argument proves that $\mathbb{N} \subset [1,\infty)$, i.e.,

$$n \geq 1,\ \ \forall n \in \mathbb{N}. \tag{3.1}$$

☆ **The Principle of Mathematical Induction.** *If E is an inductive subset of the set of natural numbers such that $1 \in E$, then $E = \mathbb{N}$.*

In applications the set E consists of the natural numbers n satisfying a property $P(n)$. To prove that any natural number n satisfies the property $P(n)$ it suffices to prove two things.

- Prove $P(1)$. This is called the *initial step.*
- Prove that if $P(n)$ is true, then so is $P(n+1)$. This is called the *inductive step.*

Sometimes we need an alternate version of the induction principle.

☆ **The Principle of Mathematical Induction: alternate version.** Suppose that for any natural number n we are given a statement $P(n)$ and we know the following.

- The statement $P(1)$ is true.
- For any $n \in \mathbb{N}$, if $P(k)$ is true for any $k < n$, then $P(n)$ is true as well.

Then $P(n)$ is true for any $n \in \mathbb{N}$. □

We will spend the rest of this section presenting various instances of the induction principle at work.

Proposition 3.4. *The sum and the product of two natural numbers are also natural numbers.*

Proof. [1] Fix a natural number m. For each $n \in \mathbb{N}$ consider the statement

$$P(n) :=\ m+n \text{ is a natural number.}$$

We have to prove that $P(n)$ is true for any $n \in \mathbb{N}$. We will achieve this using the principle of induction. We first need to check that $P(1)$ is true, i.e., that $m+1$ is a natural number. This follows from the fact that $m \in \mathbb{N}$ and $\mathbb{N}$ is an inductive set.

To complete the inductive step assume that $P(n)$ is true, i.e., $m+n \in \mathbb{N}$. Thus $(m+n)+1 \in \mathbb{N}$ and

$$m + (n+1) = (m+n)+1 \in \mathbb{N}.$$

This shows that $P(n+1)$ is also true. □

Lemma 3.5. $\forall n \in \mathbb{N}$, $(n \neq 1) \Rightarrow (n-1) \in \mathbb{N}$.

---

[1] The proof can be omitted.

Proof. For $n \in \mathbb{N}$ consider the statement

$$P(n) := n \neq 1 \Rightarrow (n-1) \in \mathbb{N}.$$

We want to prove that this statement is true for any $n \in \mathbb{N}$. The initial step is obvious since for $n = 1$ the statement $n \neq 1$ is false and thus the implication is true.

For the inductive step assume that the statement $P(n)$ is true and we prove that $P(n+1)$ is also true. Observe that $n + 1 \neq 1$ because $n \in \mathbb{N}$ and thus $n \neq 0$. Clearly $(n+1) - 1 = n \in \mathbb{N}$. □

Lemma 3.6. *The set*

$$I_1 = \{x \in \mathbb{N}; \;\; x > 1\}$$

admits a minimal element and $\min I_1 = 2$*.*

Proof. Consider the set

$$E := \{\, x \in \mathbb{N}; \;\; x = 1 \vee x \geq 2 \,\} \subset \mathbb{N}.$$

We will prove by induction that

$$E = \mathbb{N}. \tag{3.2}$$

Thus we need to show that $1 \in E$ and $x \in E \Rightarrow x + 1 \in E$. Clearly $1 \in E$.

If $x \in E$, then

- either $x = 1$ so that $x + 1 = 2 \geq 2$ so that $x + 1 \in E$,
- or $x \geq 2$ which implies $x + 1 \geq 2$ and thus $x + 1 \in E$.

The equality $E = \mathbb{N}$ implies that a natural number n is either equal to 1, or it is ≥ 2. Thus

$$x \in \mathbb{N} \wedge x > 1 \Rightarrow x \geq 2.$$

This shows that

$$x \geq 2, \;\; \forall x \in I_1.$$

Clearly $2 \in I_1$ so that $2 = \min I$. □

Corollary 3.7. *For any* $n \geq 1$ *the set*

$$H_n = \{x \in \mathbb{N}; \;\; x > n\}$$

admits a minimal element and

$$\min H_n = n + 1.$$

Proof. We will prove that for any $n \in \mathbb{N}$ the statement

$$P(n): \;\; \min H_n = n + 1$$

is true. Lemma 3.6 shows that $P(1)$ is true.

Let us show that $P(n) \Rightarrow P(n+1)$. Since $n + 2 \in H_{n+1}$ it suffices to show that $x \geq n + 2$, $\forall x \in H_{n+1}$. Let $x \in H_{n+1}$. Lemma 3.5 implies that $x - 1 \in \mathbb{N}$ and $x - 1 > n$ so that $x - 1 \in H_n$. Since $P(n)$ is true, we deduce $x - 1 \geq n + 1$, i.e., $x \geq n + 2$.

□

Corollary 3.8. *Suppose that n is a natural number. Any natural number x such that $x > n$ satisfies $x \geq n+1$.* □

Corollary 3.9. *For any natural number n, the open interval $(n, n+1)$ contains no natural number.*

Proof. From Corollary 3.8 we deduce that if x is a natural number such that $x > n$, then $x \geq n+1$. Thus there cannot exist any natural number x such that $n < x < n+1$. □

The above results imply the following important theorem.

Theorem 3.10 (Well Ordering Principle)**.** *Any set of natural numbers $S \subset \mathbb{N}$ has a minimal element.* □

For a proof of this theorem we refer to [**15**, §2.2.1].

Definition 3.11. For any $n \in \mathbb{N}$ we denote by $\mathbb{I}_n$ the set

$$\mathbb{I}_n := \big\{ x \in \mathbb{N};\ \ 1 \leq x \leq n \big\} = [1, n] \cap \mathbb{N}. \qquad \square$$

Definition 3.12. We say two sets X and Y are said to have the *same cardinality*, and we write this $X \sim Y$, if and only if there exists a bijection $f : X \to Y$. A set X is called *finite* if there exists a natural number n such that $X \sim \mathbb{I}_n$. □

Let us observe that if X, Y, Z are three sets such that $X \sim Y$ and $Y \sim Z$, then $X \sim Z$; see Exercise 3.1. This implies that any set X equivalent to a finite set Y is also finite. Indeed, if $X \sim Y$ and $Y \sim \mathbb{I}_n$, then $X \sim \mathbb{I}_n$.

At this point we want to invoke (without proof) the following result.

Proposition 3.13. *For any $m, n \in \mathbb{N}$ we have*

$$\mathbb{I}_n \sim \mathbb{I}_m \Longleftrightarrow m = n. \qquad \square$$

The above result implies that if X is a finite set, then there exists a ***unique*** natural number n such that $X \sim \mathbb{I}_n$. This unique natural number is called the *cardinality* of X and it is denoted by $|X|$ or $\#X$. You should think of the cardinality of a finite set as the number of elements in that set.

An *infinite* set is a set that is not finite. We have the following *highly nontrivial* result. Its proof is too complex to present here.

Theorem 3.14. *A set X is infinite if and only if it is equivalent to one of its* proper[2] *subsets.* □

Theorem 3.15. *The set of natural numbers $\mathbb{N}$ is infinite.*

[2]We recall that a subset $S \subset X$ is called proper if $S \neq X$.

Proof. Consider the proper subset

$$H := \{ n \in \mathbb{N};\ \ n > 1 \} \subset \mathbb{N}.$$

Lemma 3.5 implies that if $n \in H$, then $(n-1) \in \mathbb{N}$. Consider the map

$$f : H \to \mathbb{N},\ \ f(n) = n - 1.$$

Observe that this map is injective. Indeed, if $f(n_1) = f(n_2)$, then $n_1 - 1 = n_2 - 1$ so that $n_1 = n_2$. This map is also surjective. Indeed, if $m \in \mathbb{N}$. Then, according to (3.1) the natural number $n := m + 1$ is greater than 1 so it belongs to H. Clearly $f(n) = (m+1) - 1 = m$ which proves that f is also surjective. □

Definition 3.16. A set X is called *countable* if it is equivalent with the set of natural numbers. □

3.2. Applications of the induction principle

In this section we discuss some traditional applications of the induction principle. This serves two purposes: first, it familiarizes you with the usage of this principle, and second, some of the results we will discuss here will be needed later on in this class.

First let us introduce some notations. If n is a natural number, $n > 1$, and we are given n real numbers $a_1, \dots, a_n$, then define inductively

$$a_1 + \cdots + a_n := (a_1 + \cdots + a_{n-1}) + a_n,$$

$$a_1 \cdots a_n = (a_1 \cdots a_{n-1}) a_n.$$

We will use the following notations for the sum and products of a string of real numbers. Thus

$$\sum_{k=1}^{n} a_k := a_1 + \cdots + a_n,\ \ \prod_{k=1}^{n} a_k := a_1 \cdots a_n.$$

Similarly, given real numbers $a_0, a_1, \dots, a_n$ we define

$$\sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n,\ \ \prod_{k=0}^{n} a_k := a_0 \cdots a_n.$$

For any natural number n and any real number x we define inductively

$$x^n := \begin{cases} x & \text{if } n = 1 \\ (x^{n-1}) \cdot x & \text{if } n > 1. \end{cases}$$

Intuitively

$$x^n = \underbrace{x \cdot x \cdots x}_{n}.$$

If x is a *nonzero* real number we set

$$x^0 := 1.$$

Let us observe that for any natural numbers m, n and any real number x we have the equality

$$x^{m+n} = (x^m) \cdot (x^n).$$

Exercise 3.2 asks you to prove this fact.

Example 3.17. Let us prove that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \;\; \forall n \in \mathbb{N}. \tag{3.3}$$

The expanded form of the last equality is

$$1+2+\cdots+n = \frac{n(n+1)}{2}, \;\; \forall n \in \mathbb{N}.$$

Let us denote by S_n the sum $1+2+\cdots+n$. We argue by induction. The initial case $n=1$ is trivial since

$$\frac{1\cdot(1+1)}{2} = 1 = S_1.$$

For the inductive case we assume that

$$S_n = \frac{n(n+1)}{2},$$

and we have to prove that

$$S_{n+1} = \frac{(n+1)\big(\,(n+1)+1\big)}{2} = \frac{(n+1)(n+2)}{2}.$$

Indeed we have

$$S_{n+1} = S_n + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1)+2(n+1)}{2}$$

(factor out $(n+1)$)

$$= \frac{(n+1)(n+2)}{2}.$$

□

Example 3.18 (Bernoulli's inequality)**.** We want to prove a simple but very versatile inequality that goes by the name of *Bernoulli's inequality*. More precisely it states that inequality

$$\forall x \geq -1, \;\; \forall n \in \mathbb{N}: \;\; (1+x)^n \geq 1 + nx. \tag{3.4}$$

We argue by induction. Clearly, the inequality is obviously true when $n=1$ and the initial case is true. For the inductive case, we assume that

$$(1+x)^n \geq 1+nx, \;\; \forall x \geq -1 \tag{3.5}$$

and we have to prove that

$$(1+x)^{n+1} \geq 1+(n+1)x, \;\; \forall x \geq -1.$$

Since $x \geq -1$ we deduce $1+x \geq 0$. Multiplying both sides of (3.5) with the nonnegative number $1+x$ we deduce

$$(1+x)^{n+1} \geq (1+x)(1+nx) = 1+nx+x+nx^2 \geq 1+nx+x = 1+(n+1)x.$$

□

Example 3.19 (***Newton's Binomial Formula***)**.** Before we state this very important formula we need to introduce several notations widely used in mathematics. For $n \in \mathbb{N}\cup\{0\}$ we define $n!$ (read n factorial) as follows

$$0! := 1, \;\; 1! := 1, \;\; 2! = 1\cdot 2, \;\; 3! = 1\cdot 2\cdot 3, \cdots n! = 1\cdot 2\cdots n.$$

Given $k, n \in \mathbb{N}\cup\{0\}$, $k \leq n$ we define the *binomial coefficient* $\binom{n}{k}$ (read n *choose* k)

$$\binom{n}{k} := \frac{n!}{k!\cdot(n-k)!}.$$

We record below the values of these binomial coefficients for small values of n

$$\binom{0}{0}=1,\ \ \binom{1}{0}=\binom{1}{1}=1,$$

$$\binom{2}{0}=\frac{2!}{(0!)(2!)}=1,\ \ \binom{2}{1}=\frac{2!}{(1!)(1!)}=2,\ \ \binom{2}{2}=\frac{2!}{(2!)(0!)}=1,$$

$$\binom{3}{0}=\frac{3!}{(0!)(3!)}=\binom{3}{3}=1,\ \ \binom{3}{1}=\frac{(3!)}{(1!)(2!)}=\frac{3!}{(2!)(1!)}=\binom{3}{2}=3.$$

Here is a more involved example

$$\binom{7}{3}=\frac{7!}{(3!)(4!)}=\frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(3!)1\cdot 2\cdot 3\cdot 4}=\frac{7\cdot 6\cdot 5}{3!}=\frac{7\cdot 6\cdot 5}{6}=35.$$

The binomial coefficients can be conveniently arranged in the so called *Pascal triangle*

$$\begin{array}{cccccccccc}
\binom{0}{k}: & & & & & 1 & & & & \\
\binom{1}{k}: & & & & 1 & & 1 & & & \\
\binom{2}{k}: & & & 1 & & 2 & & 1 & & \\
\binom{3}{k}: & & 1 & & 3 & & 3 & & 1 & \\
\binom{4}{k}: & 1 & & 4 & & 6 & & 4 & & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Observe that each entry in the Pascal triangle is the sum of the closest neighbors above it.

The binomial coefficients play an important role in mathematics. One reason behind their usefulness is *Newton's binomial formula* which states that, for any natural number n, and any real numbers x, y, we have the equality below

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.\end{aligned} \tag{3.6}$$

We will prove this equality by induction on n. For $n=1$ we have

$$(x+y)^1=x+y=\binom{1}{0}x+\binom{1}{1}y,$$

which shows that the case $n=1$ of (3.6) is true.

As for the inductive steps, we assume that (3.6) is true for n and we prove that it is true for $n+1$. We have

$$(x+y)^{n+1}=(x+y)(x+y)^n$$

(use the inductive assumption)

$$=(x+y)\left(\binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n\right)$$

$$
\begin{aligned}
&= x\left(\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n\right)\\
&\quad + y\left(\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n\right)\\
&= \binom{n}{0}x^{n+1} + \binom{n}{1}\boxed{x^ny} + \binom{n}{2}\boxed{x^{n-1}y^2} + \cdots + \binom{n}{n-1}x^2y^{n-1} + \binom{n}{n}\boxed{xy^n}\\
&\quad + \binom{n}{0}\boxed{x^ny} + \binom{n}{1}\boxed{x^{n-1}y^2} + \binom{n}{2}x^{n-2}y^3 + \cdots + \binom{n}{n-1}\boxed{xy^n} + \binom{n}{n}y^{n+1}\\
&= \binom{n}{0}x^{n+1} + \left(\binom{n}{1} + \binom{n}{0}\right)\boxed{x^ny} + \left(\binom{n}{2} + \binom{n}{1}\right)\boxed{x^{n-1}y^2} + \cdots\\
&\quad + \left(\binom{n}{k} + \binom{n}{k-1}\right)x^{n+1-k}y^k + \cdots + \left(\binom{n}{n} + \binom{n}{n-1}\right)\boxed{xy^n} + \binom{n}{n}y^{n+1}.
\end{aligned}
$$

Clearly

$$\binom{n}{0} = 1 = \binom{n+1}{0},\quad \binom{n}{n} = 1 = \binom{n+1}{n+1}.$$

We want to show $1 \leq k \leq n$ we have the *Pascal formula*

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}. \tag{3.7}$$

Indeed, we have

$$
\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}\\
&= \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k)!(n-k+1)}\\
&= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k} + \frac{1}{n-k+1}\right)\\
&= \frac{n!}{(k-1)!(n-k)!}\cdot\left(\frac{(n-k+1)}{k(n-k+1)} + \frac{k}{k(n-k+1)}\right)\\
&= \frac{n!}{(k-1)!(n-k)!}\cdot\frac{n+1}{k(n-k+1)}\\
&= \frac{(n+1)n!}{(\,k(k-1)!\,)\cdot(\,(n-k+1)(n-k)!\,)} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}.
\end{aligned}
$$

This completes the inductive step. □

3.3. Archimedes' Principle

We begin with a simple, but fundamental observation.

Proposition 3.20. *Suppose that the nonempty subset $E \subset \mathbb{N}$ is bounded above. Then E has a maximal element n_0, i.e., $n_0 \in E$ and $n \leq n_0$, $\forall n \in E$.*

Proof. From the completeness axiom we deduce that E has a least upper bound $M = \sup E \in \mathbb{R}$. We want to prove that $M \in E$. We argue by contradiction. Suppose that $M \notin E$. In particular, this means that any number in E is strictly smaller than M.

From the definition of the *least* upper bound we deduce that there must exist $n_0 \in E$ such that

$$M - 1 < n_0 \leq M.$$

On the other hand, any natural number n greater than n_0 must be greater than or equal to $n_0 + 1$, $n \geq n_0 + 1$. Observing that $n_0 + 1 > M$, we deduce that any natural number $> n_0$ is also $> M$. Since $M \notin E$, then $n_0 < M$, and the above discussion show that the interval (n_0, M) contains no natural numbers, thus no elements of E. Hence, any real number in (n_0, M) will be an upper bound for E, contradicting that M is the *least* upper bound. □

Theorem 3.21 (Archimedes' Principle)**.** *Let ε be a positive real number. Then for any $x > 0$ there exists $n \in \mathbb{N}$ such that $n\varepsilon > x$.* [3]

Proof. Consider the set

$$E := \{ n \in \mathbb{N}; \;\; n\varepsilon \leq x \}.$$

If $E = \emptyset$, then this means that $n\varepsilon > x$ for any $n \in \mathbb{N}$ and the conclusion of the theorem is guaranteed. Suppose that $E \neq \emptyset$. Observe that

$$n \leq \frac{x}{\varepsilon}, \;\; \forall n \in E.$$

Hence, the set E is bounded above, and the previous proposition shows that it has a maximal element n_0. Then $n_0 + 1 \notin E$, so that $(n_0 + 1)\varepsilon > x$. □

Definition 3.22. The set of *integers* is the subset $\mathbb{Z} \subset \mathbb{R}$ consisting of the natural numbers, the negatives of natural numbers and 0. □

The proof of the following results are left to you as an exercise.

Proposition 3.23. *If $m, n \in \mathbb{Z}$, then $m + n, mn \in \mathbb{Z}$.* □

Proposition 3.24. *For any real number x the interval $(x-1, x]$ contains exactly one integer.* □

Corollary 3.25. *For any real number x there exists a unique integer n such that*

$$n \leq x < n + 1.$$

This integer is called the ***integer part*** *of x and it is denoted by $\lfloor x \rfloor$.*

Proof. Observe that the inequalities $n \leq x < n + 1$ are equivalent to the inequalities

$$x - 1 < n \leq x.$$

By Proposition 3.24, the interval $(x - 1, x]$ contains exactly one integer. This proves the existence and uniqueness of the integer with the postulated properties. □

[3] A popular formulation of Archimedes' principle reads: one can fill an ocean with grains of sand.

Observe for example that

$$\left\lfloor \frac{1}{2} \right\rfloor = 0, \quad \left\lfloor -\frac{1}{2} \right\rfloor = -1,$$

Theorem 3.26 (Division with remainder)**.** *Let $m, n \in \mathbb{Z}$, $n > 0$. There exists a unique pair of integers $(q, r) \in \mathbb{Z} \times \mathbb{Z}$ satisfying the following properties.*

(i) $m = qn + r$.

(ii) $0 \leq r < n$.

Proof. *Uniqueness.* Suppose that there exist two pairs of integers (q_1, r_1) and (q_2, r_2) satisfying (i) and (ii). Then

$$nq_1 + r_1 = m = nq_2 + r_2,$$

so that,

$$nq_1 - nq_2 = r_2 - r_1 \Rightarrow n(q_1 - q_2) = r_2 - r_1 \Rightarrow n \cdot |q_1 - q_2| = |r_2 - r_1|.$$

The natural numbers r_1, r_2 satisfy $0 \leq r_1, r_2 < |n|$ so that $r_1, r_2 \in [0, n-1]$. Using Exercise 2.20 we deduce $|r_2 - r_1| \leq n - 1$. Hence $n \cdot |q_1 - q_2| \leq n - 1$ which implies

$$|q_1 - q_2| \leq \frac{n-1}{n} < 1.$$

The quantity $|q_1 - q_2|$ is a nonnegative integer < 1 so that it must equal 0. This implies $q_1 = 2$ and

$$r_2 - r_1 = n(q_1 - q_2) = 0.$$

This proves the uniqueness.

Existence. Let

$$q := \left\lfloor \frac{m}{n} \right\rfloor \in \mathbb{Z}.$$

Then

$$q \leq \frac{m}{n} < q + 1 \Rightarrow nq \leq m < n(q+1) = nq + n \Rightarrow 0 \leq m - nq < n.$$

We set $r := m - nq$ and we observe that the pair (q, r) satisfies all the required properties. □

Definition 3.27. (a) Let $m, n \in \mathbb{Z}$, $m \neq 0$. We say that m *divides* n, and we write this $m|n$ if there exists an integer k such that $n = km$. When m divides n we also say that m is a *divisor* of n, or that n *is a multiple of* m, or that n *is divisible by* m.

(b) A *prime number* is a natural number $p > 1$ whose only divisors are ± 1 and $\pm p$. □

Observe that if d is a divisor of m, then $-d$ is also a divisor of m. An *even integer* is an integer divisible by 2. An *odd integer* is an integer not divisible by 2.

Given two integers m, n consider the set of common positive divisors of m and n, i.e., the set

$$D_{m,n} := \{ d \in \mathbb{N}; \ \ d|m \wedge d|n \}.$$

This set is not empty because 1 is a common positive divisor. This is bounded above because any divisor of m is $\leq |m|$. Thus the set $D_{m,n}$ has a maximal element called the *greatest common divisor* of m and n and denoted by $\gcd(m, n)$. Two integers are called *coprime* if $\gcd(m, n) = 1$, i.e., 1 is their only positive common divisor.

The next result describes on the most important property of the set $\mathbb{Z}$ of integers. We will not include its rather elaborate and tricky proof. The curious reader can find the proof in any of the books [**1**, **2**, **11**].

Theorem 3.28 (Fundamental Theorem of Arithmetic)**.** *(a) If p is a prime number that divides a product of integers mn, then $p|m$ or $p|n$.*

(b) Any natural number n can be written in a unique fashion as a product

$$n = p_1^{\alpha_1} \cdots p_k^{\alpha_k},$$

where $p_1 < p_2 < \cdots < p_k$ are prime numbers and $\alpha_1, \dots, \alpha_k$ are natural numbers. □

3.4. Rational and irrational numbers

We want to isolate another important subclasses of real numbers.

Definition 3.29. The set of *rationals* (or *rational numbers*) is the subset $\mathbb{Q} \subset \mathbb{R}$ consisting of real numbers of the form m/n where $m, n \in \mathbb{Z}$, $n \neq 0$. □

If q is a rational number, then it can be written as a fraction of the form $q = \frac{m}{n}$, $n \neq 0$. We denote by d the $\gcd(m, n)$. Thus there exist integers m_1 and n_1 such that

$$m = dm_1, \;\; n = dn_1.$$

Clearly the numbers m_1, n_1 are coprime, and we have

$$q = \frac{dm_1}{dn_1} = \frac{m_1}{n_1}.$$

We have thus proved the following result.

Proposition 3.30. *Every rational number is the ratio of two coprime integers.* □

The proof of the following result is left to you as an exercise.

Proposition 3.31. *If $q, r \in \mathbb{Q}$, then $q + r, qr \in \mathbb{Q}$.* □

We have a sequence of inclusions

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Clearly $\mathbb{N} \neq \mathbb{Z}$ because $-1 \in \mathbb{Z}$, but $-1 \notin \mathbb{N}$. Note however that, although $\mathbb{Z}$ contains $\mathbb{N}$, the set of integers $\mathbb{Z}$ is countable, i.e., it has the same cardinality as $\mathbb{N}$.

Next observe that $\mathbb{Z} \neq \mathbb{Q}$. Indeed, the rational number $1/2$ is not an integer, because it is positive and smaller than any natural number.

Similarly, although $\mathbb{Q}$ strictly contains $\mathbb{Z}$, these two sets have the same cardinality: they are both countable. However, the following **very important result** shows that, loosely speaking, there are "many more rational numbers".

Proposition 3.32 (Density of rationals)**.** *Any open interval $(a, b) \subset \mathbb{R}$, no matter how small, contains at least one rational number.*

Proof. From Archimedes' principle we deduce that there exists at least one natural number n such that $n > \frac{1}{b-a}$. Observe that $(b - a)$ is the length of the interval (a, b). This inequality is obviously equivalent to the inequality

$$\frac{1}{n} < b - a \Longleftrightarrow n(b - a) > 1$$

(This last equality codifies a rather intuitive fact: one can divide a stick of length one into many equal parts so that the subparts are as small as we please.)

We will show that we can find an integer m such that $\frac{m}{n} \in (a,b)$. Observe that

$$a < \frac{m}{n} < b \Longleftrightarrow na < m < nb \Longleftrightarrow m \in (na, nb).$$

This shows that the interval (a,b) contains a rational number if the interval (na, nb) contains an integer.

Since $n(b-a) > 1$, we deduce $nb > na + 1$. In particular, this shows that the interval $(na, na+1]$ is contained in the interval (na, nb). From Proposition 3.24 we deduce that the interval $(na, na+1]$ contains an integer m. □

This abundance of rational numbers lead people to believe for quite a long while that all real numbers must be rational. Then the ancient Greeks showed that there must exist real numbers that cannot be rational. These numbers were called *irrational*. In the remainder of this section we will describe how one can produce a large supply of irrational numbers. We start with a baby case.

Proposition 3.33. *There exists a unique positive number r such that $r^2 = 2$. This number is called the square root of 2 and it is denoted by $\sqrt{2}$*

Proof. We begin by observing the following *useful fact*:

$$\forall x, y > 0: \quad x < y \Longleftrightarrow x^2 < y^2. \tag{3.8}$$

Indeed

$$y^2 - x^2 > 0 \Longleftrightarrow (y-x)(y+x) > 0 \Longleftrightarrow y > x.$$

This useful fact takes care of the uniqueness because, if r_1, r_2 are two *positive* real numbers such that $r_1^2 = r_2^2 = 2$, then $r_1 = r_2 \Longleftrightarrow r_1^2 = r_2^2$.

To establish the existence of a positive r such that $r^2 = 2$ consider as in Example 2.16(b) the set

$$R = \big\{ x > 0; \;\; x^2 < 2 \,\big\}.$$

We have seen that this set is bounded above and thus it admits a least upper bound

$$r := \sup R.$$

We want to prove that $r^2 = 2$. We argue by contradiction and we assume that $r^2 \neq 2$. Thus, either $r^2 < 2$ or $r^2 > 2$.

Case 1. $r^2 < 2$. We will show that there exists ε_0 such that $(r + \varepsilon_0)^2 < 2$. This would imply that $r + \varepsilon_0 \in R$ and would contradict the fact that r is an upper bound for R because r would be smaller than the element $r + \varepsilon_0$ of R.

Set $\delta := 2 - r^2$. For any $\varepsilon \in (0,1)$ we have

$$(r+\varepsilon)^2 - r^2 = \big(\,(r+\varepsilon) - r\big)\big(\,(r+\varepsilon) + r\,\big) = \varepsilon(2r+\varepsilon) < \varepsilon(2r+1).$$

Now choose a number $\varepsilon_0 \in (0,1)$ such that

$$\varepsilon_0 < \frac{\delta}{2r+1}.$$

Then

$$(r+\varepsilon_0)^2 - r^2 < \varepsilon_0(2r+1) < \delta$$
$$\Rightarrow (r+\varepsilon_0)^2 < r^2 + \delta = r^2 + 2 - r^2 = 2 \Rightarrow r + \varepsilon_0 \in R.$$

Case 2. $r^2 > 2$. We will prove that under this assumption

$$\exists \varepsilon_0 \in (0,1) \ \text{ such that } \ r - \varepsilon_0 > 0 \ \text{ and } \ (r-\varepsilon_0)^2 > 2. \tag{3.9}$$

Let us observe that (3.9) leads to a contradiction. Indeed, observe that $(r-\varepsilon_0)$ is an upper bound for R. Indeed, if $x \in R$, then

$$x^2 < 2 < (r-\varepsilon_0)^2 \overset{(3.8)}{\Rightarrow} x < r_0 - \varepsilon.$$

Thus, $r - \varepsilon_0$ is an upper bound of R and this upper bound is obviously strictly smaller than r, the *least* upper bound of R. This is a contradiction which shows that the situation $r^2 > 2$ is also not possible. Let us now prove (3.9).

Denote by δ the difference $\delta = r^2 - 2 > 0$. For any $\varepsilon \in (0,r)$ we have

$$r^2 - (r-\varepsilon)^2 = \Big(r - (r-\varepsilon)\Big)\Big(r + (r-\varepsilon)\Big) = \varepsilon(2r-\varepsilon) < 2r\varepsilon.$$

We have thus shown that for any $\varepsilon \in (0,r)$ we have $(r-\varepsilon) > 0$ and

$$r^2 - (r-\varepsilon)^2 \leq 2r\varepsilon \Longleftrightarrow (r-\varepsilon)^2 \geq r^2 - 2r\varepsilon.$$

Now choose $\varepsilon_0 \in (0,r)$ small enough so that $\varepsilon_0 < \frac{\delta}{2r}$. Hence $2r\varepsilon_0 < \delta$ so that $-2r\varepsilon_0 > -\delta$ and

$$(r-\varepsilon_0)^2 > r^2 - 2r\varepsilon_0 > r^2 - \delta = r^2 - (r^2 - 2) = 2.$$

We deduce again that the situation $r^2 > 2$ is not possible so that $r^2 = 2$.

□

The result we have just proved can be considerably generalized.

Theorem 3.34. *Fix a natural number $n \geq 2$. Then for any positive real number a* there exists a unique, positive *real number r such that $r^n = a$.*

Proof. *Existence.* Consider the set

$$S := \big\{ s \in \mathbb{R};\ \ s \geq 0 \wedge s^n \leq a \big\}.$$

Observe that this is a nonempty set since $0 \in S$. We want to prove that S is also bounded. To achieve this we need a few auxiliary results.

Lemma 3.35 (**A very handy identity**)**.** *For any real numbers x, y and any natural number n we have the equality*

$$x^n - y^n = \big(x - y\big) \cdot \big(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\big) \tag{3.10}$$

Proof. We have

$$\big(x-y\big)\cdot\big(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\big)$$
$$= x\big(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\big) - y\big(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}\big)$$
$$= x^n + x^{n-1}y + x^{n-2}y^2 + \cdots + x^2y^{n-2} + xy^{n-1}$$
$$- x^{n-1}y - x^{n-2}y^2 - \cdots - x^2y^{n-2} - xy^{n-1} - y^n$$

$$= x^n - y^n.$$

□

Here is an immediate useful consequence of this identity.

$$\forall n \in \mathbb{N},\ \ \forall x, y > 0:\ \ x < y \Longleftrightarrow x^n < y^n. \tag{3.11}$$

Indeed

$$y^n - x^n > 0 \Longleftrightarrow (y-x)(y^n + y^{n-2}x + \cdots + x^n) > 0 \Longleftrightarrow y - x > 0.$$

Lemma 3.36. *Any positive real number x such that $x^n \geq a$ is an upper bound for S. In particular, any natural number $k > a$ is an upper bound for S so that S is a bounded set.*

Proof. Let x be a positive real number such that $x^n \geq a$. We want to prove that $x \geq s$ for any $s \in S$. Indeed

$$s \in S \Rightarrow s^n \leq a \leq x^n \overset{(3.11)}{\Rightarrow} s \leq x.$$

This proves the first part of the lemma.

Suppose now that k is a natural number such that $k > a$. Observe first that

$$k^n > k^{n-1} > \cdots > k > a.$$

From the first part of the lemma we deduce that k is an upper bound for S. □

The nonempty set S is bounded above. The Completeness Axiom implies that it admits a least upper bound

$$r := \sup S.$$

We will show that $r^n = a$. We argue by contradiction and we assume that $r^n \neq a$. Thus, either $r^n < a$, or $r^n > a$.

Case 1. $r^n < a$. We will show that we can find $\varepsilon_0 \in (0,1)$ such that $(r + \varepsilon_0)^n < a$. This would imply that $r + \varepsilon_0 \in S$ and it would contradict the fact that r is an upper bound for S because r is less than the number $r + \varepsilon_0 \in S$.

Denote by δ the difference $\delta := a - r^n > 0$. For any $\varepsilon \in (0,1)$ we have

$$(r+\varepsilon)^n - r^n = \Big((r+\varepsilon) - r \Big)\Big((r+\varepsilon)^{n-1} + (r+\varepsilon)^{n-2}r^2 + \cdots + r^{n-1} \Big)$$

$(r + \varepsilon < r + 1)$

$$\leq \varepsilon \underbrace{\Big((r+1)^{n-1} + (r+1)^{n-2}r + \cdots + r^{n-1} \Big)}_{=:q}$$

We have thus proved that

$$(r+\varepsilon)^n \leq r^n + \varepsilon q,\ \ \forall \varepsilon \in (0,1).$$

Choose $\varepsilon_0 \in (0,1)$ small enough so that

$$\varepsilon_0 < \frac{\delta}{q} \Longleftrightarrow \varepsilon_0 q < \delta.$$

Then

$$(r+\varepsilon_0)^n \leq r^n + \varepsilon_0 q < r^n + \delta = a \Rightarrow r + \varepsilon_0 \in S.$$

This contradicts the fact that r is an upper bound for S and shows that the inequality $r^n < a$ is impossible.

Case 2. $r^n > a$. We will prove that under this assumption

$$\exists \varepsilon_0 \in (0,1) \text{ such that } r - \varepsilon_0 > 0 \text{ and } (r-\varepsilon_0)^n > a. \tag{3.12}$$

Let us observe that (3.12) leads to a contradiction. Indeed, Lemma 3.36 implies that $r - \varepsilon_0$ is an upper bound of S and this upper bound is obviously strictly smaller than r, the *least* upper bound of S. This is a contradiction which shows that the situation $b^n > a$ is also not possible. Let us now prove (3.12).

Denote by δ the difference $\delta = r^n - a > 0$. For any $\varepsilon \in (0,r)$ we have

$$r^n - (r-\varepsilon)^n = \Big(r - (r-\varepsilon)\Big)\Big(r^{n-1} + r^{n-2}(r-\varepsilon) + \cdots + \cdots + (r-\varepsilon)^{n-1}\Big)$$

$((r-\varepsilon) < b)$

$$\leq \varepsilon \underbrace{(r^{n-1} + r^{n-2}r + \cdots + r^{n-1})}_{=:q}.$$

We have thus shown that for any $\varepsilon \in (0,r)$ we have $(r-\varepsilon) > 0$ and

$$r^n - (r-\varepsilon)^n \leq \varepsilon q \Longleftrightarrow (r-\varepsilon)^n \geq r^n - \varepsilon q.$$

Now choose $\varepsilon_0 \in (0,c)$ small enough so that $\varepsilon_0 < \frac{\delta}{q}$. Hence $\varepsilon_0 q < \delta$ so that $-\varepsilon_0 q > -\delta$ and

$$(r-\varepsilon_0)^n > r^n - \varepsilon_0 q > r^n - \delta = r^n - (r^n - a) = a.$$

We deduce again that the situation $r^n > a$ is not possible so that $r^n = a$. This completes the existence part of the proof.

Uniqueness. Suppose that r_1, r_2 are two positive numbers such that $r_1^n = r_2^n = a$. Using (]3.11) we deduce that $r_1 = r_2$.This completes the proof of Theorem 3.34. □

The above result leads to the following important concept.

Definition 3.37. Let a be a positive real number and $n \in \mathbb{N}$. The *n-th root* of a, denoted by $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$ is the ***unique positive real number*** r such that $r^n = a$. □

Theorem 3.38. *The positive number $\sqrt{2}$ is not rational.*

Proof. We argue by contradiction and we assume that $\sqrt{2}$ is rational. It can therefore be represented as a fraction,

$$\sqrt{2} = \frac{m}{n}, \quad m,n \in \mathbb{N}, \quad \gcd(m,n) = 1.$$

Thus $2 = \frac{m^2}{n^2}$ and we deduce

$$2n^2 = m^2. \tag{3.13}$$

Since 2 is a prime number and $2|m^2$ we deduce that $2|m$, i.e., $m = 2m_1$ for some natural number m_1. Using this last equality in (3.13) we deduce

$$2n^2 = (2m_1)^2 = 4m_1^2 \Rightarrow n^2 = 2m_1^2.$$

Thus $2|n^2$, and arguing as above we deduce that $2|n$. Hence 2 is a common divisor of both m and n. This contradicts the starting assumption that $\gcd(m,n) = 1$ and proves that $\sqrt{2}$ cannot be rational. □

Now that we know that there exist irrational numbers, we can ask, how many are they. It turns out that most real numbers are irrational, but we will not prove this fact now.

3.5. Exercises

Exercise 3.1. (a) Suppose that X, Y are two sets such that $X \sim Y$. Prove that $Y \sim X$.
(b) Prove that if X, Y, Z are sets such that $X \sim Y$ and $Y \sim Z$, then $X \sim Z$. $\square$

Exercise 3.2. Prove by induction that for any natural numbers m, n and any real number x we have the equality

$$x^{m+n} = (x^m) \cdot (x^n). \qquad \square$$

Exercise 3.3. (a) Prove that for any natural number n and any real numbers

$$a_1, a_2, \dots, a_n, b_1, \dots, b_n, c$$

we have the equalities

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{i=1}^{n} a_i + \sum_{j=1}^{n} b_j, \quad \sum_{k=1}^{n}(ca_k) = c\left(\sum_{k=1}^{n} a_k\right). \qquad \square$$

(b) Using (a) and (3.3) prove that for any natural number n and any real numbers a, r we have the equality

$$\sum_{k=0}^{n}(a + kr) = a + (a + r) + (a + 2r) \cdots + (a + nr) = (n + 1)a + \frac{rn(n + 1)}{2}.$$

(c) Use (b) to compute

$$3 + 7 + 11 + 15 + 19 + \cdots + 999,999.$$

Express the above using the symbol $\sum$.
(d) Prove that for any natural number n we have the equality

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2. \qquad \square$$

Exercise 3.4. Prove that for any natural number n and any positive real numbers x, y such that $x < 1 < y$ we have

$$x^n \leq x, \;\; y \leq y^n. \qquad \square$$

Exercise 3.5. Prove that if $0 < a < b$, and $n \geq 2$, then

$$\sqrt[n]{a} < \sqrt[n]{b}, \;\; a < \sqrt{ab} < \frac{a + b}{2} < b. \qquad \square$$

Exercise 3.6. Find a natural number N_0 with the following property: for any $n > N_0$ we have

$$0 < \frac{n}{n^2 + 1} < \frac{1}{10^6} = \frac{1}{1,000,000}. \qquad \square$$

Exercise 3.7. Prove that for any natural number n and any real number $x \neq 1$ we have the equality.

$$\frac{1 - x^n}{1 - x} = 1 + x + x^2 + \cdots + x^{n-1}. \qquad \square$$

Exercise 3.8. (a) Compute

$$\binom{11}{2},\ \binom{11}{3},\ \binom{11}{8},\ \binom{15}{4},\ \binom{15}{11}.$$

(b) Show that for any $n, k \in \mathbb{N} \cup \{0\}$, $k \leq n$ we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

(c) Use Newton's binomial formula to show that for any natural number n we have the equalities

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n,$$

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots + (-1)^n\binom{n}{n} = 0.$$

Deduce that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}.$$

□

Exercise 3.9. Show that for any real number x, the interval $(x-1, x]$ contains exactly one integer.

Hint: For uniqueness use the Corollaries 3.8 and 3.9. To prove existence consider separately the cases

- $x \in \mathbb{Z}$.
- $(x \in \mathbb{R} \setminus \mathbb{Z}) \wedge (x > 0)$.
- $(x \in \mathbb{R} \setminus \mathbb{Z}) \wedge (x < 0)$.

□

Exercise 3.10. Let a, b, c be real numbers, $a \neq 0$.

(a) Show that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a}\right),\quad \forall x \in \mathbb{R}.$$

(b) Prove that the following statements are equivalent.

(i) There exist $r_1, r_2 \in \mathbb{R}$ such that

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

(ii) There exists $r \in \mathbb{R}$ such that $ar^2 + br + c = 0$.

(iii) $b^2 - 4ac \geq 0$.

□

Exercise 3.11. Find the ranges of the functions

$$f : (-\infty, 5) \to \mathbb{R},\quad f(x) = \frac{x+1}{x-5},$$

and

$$g:\mathbb{R}\to\mathbb{R},\;\; g(x)=\frac{x}{x^2+1}. \qquad \square$$

Exercise 3.12. (a) Show that the equation $x^2-x-1=0$ has two solutions $r_1,r_2\in\mathbb{R}$ and then prove that r_1,r_2 satisfy the equalities

$$r_1+r_2=1,\;\; r_1r_2=-1.$$

(b) For any nonnegative integer n we set

$$F_n=\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2},$$

where r_1,r_2 are as in (a). Compute F_0,F_1,F_2.
(c) Prove by induction that for any nonnegative integer n we have

$$r_1^{n+2}=r_1^{n+1}+r_1^n,\;\; r_2^{n+2}=r_2^{n+1}+r_2^n,$$

and

$$F_{n+2}=F_{n+1}+F_n.$$

(d) Use the above equality to compute $F_3,\dots,F_9$. $\square$

Exercise 3.13. Prove Propositions 3.23 and 3.31 . $\square$

Exercise 3.14. (a) Verify that for any $a,b>0$ and any $m,n\in\mathbb{N}$ we have the equalities

$$(ab)^{\frac{1}{n}}=a^{\frac{1}{n}}\cdot b^{\frac{1}{n}},\;\; \bigl(a^{\frac{1}{n}}\bigr)^{\frac{1}{m}}=a^{\frac{1}{mn}},$$

$$(a^m)^{\frac{1}{n}}=\bigl(a^{\frac{1}{n}}\bigr)^m=:a^{\frac{m}{n}},$$

$$a^{\frac{km}{kn}}=a^{\frac{m}{n}},\;\; \forall k\in\mathbb{N}.$$

$$(a^{\frac{m}{n}})^{-1}=\bigl(a^{-1}\bigr)^{\frac{m}{n}}=:a^{-\frac{m}{n}}.$$

$$a^{-\frac{km}{kn}}=a^{-\frac{m}{n}},\;\; \forall k\in\mathbb{N}.$$

☞ *Recall that an expression of the form " bla-bla-bla =: x" signifies that the quantity x is defined to be whatever bla-bla-bla means. In particular the notation*

$$\bigl(a^{\frac{1}{n}}\bigr)^m=:a^{\frac{m}{n}}$$

indicates that the quantity $a^{\frac{m}{n}}$ is defined to be the m-th power of the n-th root of a.
(b) Prove that if $a>0$, then for any $m,m'\in\mathbb{Z}$ and $n,n'\in\mathbb{N}$ such that

$$\frac{m}{n}=\frac{m'}{n'},$$

then

$$a^{\frac{m}{n}}=a^{\frac{m'}{n'}}$$

Any rational number r admits a nonunique representation as a fraction

$$r=\frac{m}{n},\;\; m\in\mathbb{Z},\;\; n\in\mathbb{N}.$$

Part (b) allows us to give a well defined meaning to a^r, >0, $r\in\mathbb{Q}$.

(c) Show that for all $r_1, r_2 \in \mathbb{Q}$ and any $a > 0$ we have

$$a^{r_1} \cdot a^{r_2} = a^{r_1 + r_2}.$$

(d) Suppose that $a > b > 0$. Prove that for any rational number $r > 0$ we have

$$a^r > b^r.$$

(e) Suppose that $a > 1$. Prove that for any rational numbers r_1, r_2 such that $r_1 < r_2$ we have

$$a^{r_1} < a^{r_2}.$$

(f) Suppose that $a \in (0, 1)$. Prove that for any rational numbers r_1, r_2 such that $r_1 < r_2$ we have

$$a^{r_1} > a^{r_2}.$$

□

3.6. Exercises for extra-credit

Exercise* 3.1. Consider the map $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = (-1)^{n+1} \left\lfloor \frac{n}{2} \right\rfloor .$$

(i) Compute $f(1), f(2), f(3), f(4), f(5), f(6), f(7)$.

(ii) Given a natural number k, compute $f(2k)$ and $f(2k-1)$.

(iii) Prove that f is a bijection.

□

Exercise* 3.2. (a) Let p be a prime number and n a natural number > 1. Prove that $\sqrt[n]{p}$ is irrational.

(b) Let m, n be natural numbers and p, q prime numbers. Prove that

$$p^{1/m} = q^{1/n} \iff (p = q) \wedge (m = n).$$ □

Exercise* 3.3. Let $S \subset [0, 1]$ be a set satisfying the following two properties.

(i) $0, 1 \in S$.

(ii) For any $n \in \mathbb{N}$ and any pairwise distinct numbers $s_1, \dots, s_n \in S$ we have

$$\frac{s_1 + \cdots + s_n}{n} \in S.$$

Show that $S = \mathbb{Q} \cap [0, 1]$. □

Exercise* 3.4. Given 25 positive real numbers, prove that you can choose two of them x, y so none of the remaining numbers is equal to the sum $x + y$ or the differences $x - y$, $y - x$.□

Exercise* 3.5. At a stockholder' meeting, the board presents the month-by-month profit (or losses) since the last meeting. "Note" says the CEO, "that we made a profit over every consecutive eight-month period."

"Maybe so", a shareholder complains, "but I also see that we *lost* over every consecutive *five*-month period!"

What is the maximum number of months that could have passed since the last meeting? □

Exercise* 3.6 (Erdös-Szekeres)**.** Suppose we are given an injection $f : \{1, \dots, 10001\} \to \mathbb{R}$. Prove that there exists a subset $I \subset \{1, \dots, 10001\}$ of cardinality 101 such that, either

$$f(i_1) < f(i_2), \;\; \forall i_1, i_2 \in I, \;\; i_1 < i_2,$$

or

$$f(i_1) > f(i_2), \;\; \forall i_1, i_2 \in I, \;\; i_1 < i_2. \qquad \square$$

Exercise* 3.7 (Chebyshev)**.** Suppose that $p_1, \dots, p_n$ are positive numbers such that

$$p_1 + \cdots + p_n = 1.$$

Prove that if $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are real numbers such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \;\text{ and }\; y_1 \leq y_2 \leq \cdots \leq y_n,$$

then

$$\sum_{k=1}^{n} x_k y_k p_k \geq \left(\sum_{i=1}^{n} x_i p_i \right) \left(\sum_{j=1}^{n} y_j p_j \right). \qquad \square$$

Exercise* 3.8. Let $k \in \mathbb{N}$. We are given k pairwise disjoint intervals $I_1, \dots, I_k \subset [0, 1]$. Denote by S their union. We know that for any $d \in [0, 1]$ there exist two points $p, q \in S$ such that $\operatorname{dist}(p, q) = d$. Prove that

$$\text{length}\,(I_1) + \cdots + \text{length}\,(I_k) \geq \frac{1}{k}. \qquad \square$$

Chapter 4

Limits of sequences

The concept of limit is the central concept of this course. This chapter deals with the simplest incarnation of this concept, namely the notion of limit of a sequence of real numbers.

4.1. Sequences

Formally, a *sequence of real numbers* is a function $x : \mathbb{N} \to \mathbb{R}$. We typically describe a sequence $x : \mathbb{N} \to \mathbb{R}$ as a list $(x_n)_{n\in\mathbb{N}}$ consisting of one real number for each natural number n,

$$x_1, x_2, x_3, \dots, x_n, \dots .$$

Often we will allow lists that start at time 0, $(x_n)_{n\geq 0}$,

$$x_0, x_1, x_2, \dots .$$

If we use our intuition of a real number as corresponding to a point on a line, we can think of a sequence $(x_n)_{n\geq 1}$ as describing the motion of an object along the line, where x_n describes the position of that object at time n.

Example 4.1. (a) The natural numbers form a sequence $(n)_{n\in\mathbb{N}}$,

$$1, 2, 3, \dots .$$

(b) The *arithmetic progression* with initial term $a \in \mathbb{R}$ and ratio $r \in \mathbb{R}$ is the sequence

$$a, a + r, a + 2r, a + 3r, \dots .$$

For example, the sequence

$$3, 7, 11, 15, 19, \cdots$$

is an arithmetic progression with initial term 3 and ratio 4. The constant sequence

$$a, a, a, \dots ,$$

is an arithmetic progression with initial term 0 and ratio 0.

(c) The *geometric progression* with initial term $a \in \mathbb{R}$ and ratio $r \in \mathbb{R}$ is the sequence

$$a, ar, ar^2, ar^3, \dots .$$

For example, the sequence

$$1, -1, 1, -1,$$

is the geometric progression with initial term 1 and ratio -1.

(d) The *Fibonacci sequence* is the sequence $F_0, F_1, F_2, \dots$ given by the initial condition

$$F_0 = F_1 = 1,$$

and the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, \;\; \forall n \geq 0.$$

For example

$$F_2 = 1 + 1 = 2, \;\; F_3 = 2 + 1 = 3, \;\; F_4 = 3 + 2 = 5, \;\; F_5 = 5 + 3 = 8, \dots .$$

In Exercise 3.12 we gave an alternate description to the Fibonacci sequence. □

Definition 4.2. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

(i) The sequence $(x_n)_{n\in\mathbb{N}}$ is called *increasing* if

$$x_n < x_{n+1}, \;\; \forall n \in \mathbb{N}.$$

(ii) The sequence $(x_n)_{n\in\mathbb{N}}$ is called *decreasing* if

$$x_n > x_{n+1}, \;\; \forall n \in \mathbb{N}.$$

(iii) The sequence $(x_n)_{n\in\mathbb{N}}$ is called *non-increasing* if

$$x_n \geq x_{n+1}, \;\; \forall n \in \mathbb{N}.$$

(iv) The sequence $(x_n)_{n\in\mathbb{N}}$ is called *non-decreasing* if

$$x_n \leq x_{n+1}, \;\; \forall n \in \mathbb{N}.$$

(v) A sequence $(x_n)_{n\in\mathbb{N}}$ is called *monotone* if it is either non-decreasing, or non-increasing. It is called *strictly monotone* if it is either increasing, or decreasing.

(vi) The sequence $(x_n)_{n\in\mathbb{N}}$ is called *bounded* if there exist real numbers m, M such that

$$m \leq x_n \leq M, \;\; \forall n \in \mathbb{N}.$$ □

Note that an arithmetic progression is increasing if and only if its ratio is positive, while a geometric progression with positive initial term and positive ratio is monotone: it is increasing if the ratio is > 1, decreasing if the ratio < 1 and constant if the ratio is $= 1$. A geometric progression is bounded if and only if its ratio r satisfies $|r| \leq 1$.

A *subsequence* of a sequence $x : \mathbb{N} \to \mathbb{R}$ is a restriction of x to an infinite subset $S \subset \mathbb{N}$. An infinite subset $S \subset \mathbb{N}$ can itself be viewed as an *increasing* sequence of natural numbers

$$n_1 < n_2 < n_3 < \dots,$$

where

$$n_1 := \min S, \;\; n_2 = \min S \setminus \{n_1\}, \dots, n_{k+1} := \min S \setminus \{n_1, \dots, n_k\}, \dots .$$

Thus a subsequence of a sequence $(x_n)_{n\in\mathbb{N}}$ can be described as a sequence $(x_{n_k})_{k\in\mathbb{N}}$, where $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence of natural numbers.

4.2. Convergent sequences

Definition 4.3. We say that the sequence of real numbers (x_n) *converges to the number* $x \in \mathbb{R}$ if

$$\forall \varepsilon > 0: \ \exists N = N(\varepsilon) \in \mathbb{N} \text{ such that } \forall n > N(\varepsilon) \text{ we have } |x_n - x| < \varepsilon. \tag{4.1}$$

A sequence (x_n) is called *convergent* if it converges to some number x. More precisely, this means

$$\exists x \in \mathbb{R}, \ \forall \varepsilon > 0: \ \exists N = N(\varepsilon) \in \mathbb{N} \text{ such that } \forall n > N(\varepsilon) \text{ we have } |x_n - x| < \varepsilon. \tag{4.2}$$

The number x is called *a limit* of the sequence (a_n). A sequence is called *divergent* if it is not convergent. □

Observe that condition (4.1) can be rephrased as follows

$$\forall \varepsilon > 0: \ \exists N = N(\varepsilon) \in \mathbb{N} \text{ such that } \forall n > N(\varepsilon) \text{ we have } \operatorname{dist}(x_n, x) < \varepsilon. \tag{4.3}$$

Before we proceed further, let us observe the following simple fact.

Proposition 4.4. *Given a sequence (x_n) there exists at most one real number x satisfying the convergence property (4.1).*

Proof. Suppose that x, x' are two real numbers satisfying (4.1). Thus,

$$\forall \varepsilon > 0: \ \exists N = N(\varepsilon) \in \mathbb{N} \text{ such that } \forall n > N \text{ we have } |x_n - x| < \varepsilon,$$

and

$$\forall \varepsilon > 0: \ \exists N' = N'(\varepsilon) \in \mathbb{N} \text{ such that } \forall n > N', \text{ we have } |x_n - x'| < \varepsilon.$$

Thus, if $n > N_0(\varepsilon) := \max(\,N(\varepsilon), N'(\varepsilon)\,)$ then

$$|x_n - x|, \ |x_n - x'| < \varepsilon.$$

We observe that if $n > N_0(\varepsilon)$, then

$$|x - x'| = |(x - x_n) + (x_n - x')| \leq |x - x_n| + |x_n - x'| < 2\varepsilon.$$

In other words

$$\forall \varepsilon > 0: \ |x - x'| < 2\varepsilon, \ \forall n > N_0(\varepsilon).$$

In the above statement the variable n really plays no role: if $|x - x'| < 2\varepsilon$ for some n, then clearly $|x - x'| < 2\varepsilon$ for any n. We conclude that

$$\forall \varepsilon > 0: \ |x - x'| < 2\varepsilon.$$

In other words, the distance $\operatorname{dist}(x, x') = |x - x'|$ between x and x' is smaller than any positive real number, so that this distance must be zero (Exercise 2.14) and hence $x = x'$. □

Definition 4.5. Given a convergent sequence (x_n), the unique real number x satisfying the convergence condition (4.1) is called *the limit* of the sequence (x_n) and we will indicate this using the notations

$$x = \lim_{n \to \infty} x_n \ \text{ or } \ x = \lim_n x_n.$$

We will also say that (x_n) *tends (or converges) to x as n goes to ∞.* □

Observe that

$$\lim_{n\to\infty} x_n = x \Longleftrightarrow \lim_{n\to\infty} |x_n - x| = 0. \tag{4.4}$$

The next example shows that convergent sequences do exist.

Example 4.6. (a) If (x_n) is the constant sequence, $x_n = x$, for all n, then (x_n) is convergent and its limit is a.

(b) We want to show that

$$\boxed{\lim_{n\to\infty} \frac{C}{n} = 0,\ \ \forall C > 0}. \tag{4.5}$$

Let $\varepsilon > 0$ and set $N(\varepsilon) := \left\lfloor \frac{C}{\varepsilon} \right\rfloor + 1 \in \mathbb{N}$. We deduce

$$N(\varepsilon) > \frac{C}{\varepsilon},\ \text{ i.e., }\ \frac{N(\varepsilon)}{C} > \frac{1}{\varepsilon}.$$

For any $n > N(\varepsilon)$ we have

$$\frac{n}{C} > \frac{N(\varepsilon)}{C} > \frac{1}{\varepsilon} \Rightarrow \frac{C}{n} < \varepsilon.$$

Hence for any $n > N(\varepsilon)$ we have

$$|x_n| = \frac{C}{n} < \varepsilon.$$

□

Definition 4.7. (a) A *neighborhood* of a real number x is defined to be an *open* interval (α, β) that contains x, i.e., $x \in (\alpha, \beta)$.

(b) A *neighborhood of* ∞ is an interval of the form (M, ∞), while a *neighborhood of* $-\infty$ is an interval of the form $(-\infty, M)$. □

We have the following equivalent description of convergence. Its proof is left to you as an exercise.

Proposition 4.8. *Let (x_n) be a sequence of real numbers. Prove that the following statements are equivalent.*

(i) *The sequence (x_n) converges to $x \in \mathbb{R}$ as $n \to \infty$.*

(ii) *For any neighborhood U of x there exists a natural number N such that*

$$\forall n \big(n > N \Rightarrow x_n \in U \big).$$

□

The proof of the following result is left to you as an exercise.

Proposition 4.9. *Suppose that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence and $x = \lim_{n\to\infty} x_n$.*

(i) *If $(x_{n_k})_{k\geq 1}$ is a subsequence of (x_n), then*

$$\lim_{k\to\infty} x_{n_k} = x.$$

(ii) *Suppose that $(x'_n)_{n\in\mathbb{N}}$ is another sequence with the following property*

$$\exists N_0 \in \mathbb{N}:\ \ \forall n > N_0\ \ x'_n = x_n.$$

Then

$$\lim_{n\to\infty} x'_n = x.$$

□

Part (ii) of the above proposition shows that the convergence or divergence of a sequence is not affected if we modify only finitely many of its terms. The next result is very intuitive.

Proposition 4.10 (Squeezing Principle)**.** *Let (a_n) (x_n), (y_n) be sequences such that*

$$\exists N_0 \in \mathbb{N} : \forall n > N_0, \;\; x_n \leq a_n \leq y_n.$$

If

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a,$$

then

$$\lim_{n\to\infty} a_n = a.$$

Proof. We have

$$\operatorname{dist}(a_n, a) \leq \operatorname{dist}(a_n, x_n) + \operatorname{dist}(x_n, a).$$

Since a_n lies in the interval $[x_n, y_n]$ for $n > N_0$ we deduce that

$$\operatorname{dist}(a_n, x_n) \leq \operatorname{dist}(y_n, x_n), \;\; \forall n > N_0,$$

so that

$$\operatorname{dist}(a_n, a) \leq \operatorname{dist}(y_n, x_n) + \operatorname{dist}(x_n, a), \;\; \forall n > N_0.$$

Now observe that

$$\operatorname{dist}(y_n, x_n) \leq \operatorname{dist}(y_n, a) + \operatorname{dist}(a, x_n).$$

Hence,

$$\begin{aligned} \operatorname{dist}(a_n, a) &\leq \operatorname{dist}(y_n, a) + \operatorname{dist}(a, x_n) + \operatorname{dist}(x_n, a) \\ &= \operatorname{dist}(y_n, a) + 2\operatorname{dist}(x_n, a), \;\; \forall n > N_0. \end{aligned} \tag{4.6}$$

Let $\varepsilon > 0$. Since $x_n \to a$ there exists $N_x(\varepsilon) \in \mathbb{N}$ such that

$$\forall n > N_x(\varepsilon) : \;\; \operatorname{dist}(x_n, a) < \frac{\varepsilon}{3}.$$

Since $y_n \to a$ there exists $N_y(\varepsilon) \in \mathbb{N}$ such that

$$\forall n > N_y(\varepsilon) : \;\; \operatorname{dist}(y_n, a) < \frac{\varepsilon}{3}.$$

Set $N(\varepsilon) := \max\{N_0, N_x(\varepsilon), N_y(\varepsilon)\}$. For $n > N(\varepsilon)$ we have

$$\operatorname{dist}(x_n, a) < \frac{\varepsilon}{3}, \;\; \operatorname{dist}(y_n, a) < \frac{\varepsilon}{3}$$

and thus

$$\operatorname{dist}(y_n, a) + 2\operatorname{dist}(x_n, a) < \varepsilon.$$

Using this in (4.6) we conclude that

$$\forall n > N(\varepsilon) \;\; \operatorname{dist}(a_n, a) < \varepsilon.$$

This proves that $a_n \to a$ as $n \to \infty$. □

Corollary 4.11. *Suppose that $a \in \mathbb{R}$ and (a_n), (x_n) are sequences of real numbers such that*

$$|a_n - a| \le x_n \;\; \forall n, \;\; \lim_{n\to\infty} x_n = 0.$$

Then

$$\lim_{n\to\infty} a_n = a.$$

Proof. We have squeezed the sequence $|a_n - a|$ between the sequences (x_n) and the constant sequence 0, both converging to 0. Hence $|a_n - a| \to 0$ and, in view of (4.4), we deduce that also $a_n \to a$. □

Example 4.12. We want to show that

$$\boxed{\forall M > 0, \;\; \forall r \in (-1,1) \lim_{n\to\infty} M r^n = 0}. \tag{4.7}$$

Clearly, it suffices to show that $M|r|^n \to 0$. This is clearly the case if $r = 0$. Assume $r \ne 0$. Set

$$R := \frac{1}{|r|}.$$

Then $R > 1$ so that $R = 1 + \delta$, $\delta > 0$. Bernoulli's inequality (3.4) implies that $\forall n \in \mathbb{N}$ we have $R^n \ge 1 + n\delta$ so that

$$M|r|^n = \frac{M}{R^n} \le \frac{M}{1+n\delta} \le \frac{M}{n\delta} = \frac{C}{n}, \;\; C := \frac{M}{\delta}.$$

From Example 4.6 (b) we deduce that

$$\lim_n \frac{C}{n} = 0.$$

The desired conclusion now follows from the Squeezing Principle. □

Example 4.13. We want to prove that

$$\boxed{\lim_n \frac{r^n}{n!} = 0, \;\; \forall r \in \mathbb{R}}. \tag{4.8}$$

We will rely again on the Squeezing Principle. Fix $N_0 \in \mathbb{N}$ such that $N_0 > 2|r|$. Then for any $n > N_0$ we have

$$\left|\frac{r^n}{n!}\right| = \frac{|r|^n}{n!} = \frac{|r|^{N_0} r^{n-N_0}}{1\cdot 2\cdots N_0\cdot (N_0+1)(N_0+2)\cdots n}$$

$$= \underbrace{\frac{|r|^{N_0}}{N_0!}}_{=:C_0}\cdot \underbrace{\frac{|r|}{N_0+1}\cdot\frac{|r|}{N_0+2}\cdots\frac{|r|}{n}}_{(n-N_0)\text{ terms}}.$$

Now observe that

$$\frac{|r|}{N_0+1}, \frac{|r|}{N_0+2}, \dots, \frac{|r|}{n} < \frac{|r|}{N_0} < \frac{1}{2},$$

and we deduce

$$\left|\frac{r^n}{n!}\right| < C_0\left(\frac{1}{2}\right)^{n-N_0} = C_0\left(\frac{1}{2}\right)^{-N_0}\left(\frac{1}{2}\right)^{n} = 2^{N_0} C_0 2^{-n}.$$

If we denote by M the constant $2^{N_0}C_0$ and we set $x_n := M2^{-n}$, $n \in \mathbb{N}$, we deduce that

$$\forall n > N_0 : \quad \left|\frac{r^n}{n!}\right| < x_n.$$

Example 4.12 shows that $x_n \to 0$ and the conclusion (4.8) now follows from the Squeezing Principle. □

Proposition 4.14. *Any convergent sequence of real numbers is bounded.*

Proof. Suppose that $(a_n)_{n\geq 1}$ is a convergent sequence

$$a = \lim_{n\to\infty} a_n.$$

There exists $N \in \mathbb{N}$ such that, for any $n > N$ we have

$$|a_n - a| < 1.$$

Thus, for any $n > N$ we have $a_n \in (a-1, a+1)$. Now set

$$m := \min\bigl\{a_1, a_2, \dots, a_N, a-1\bigr\}, \;\; M := \max\bigl\{a_1, a_2, \dots, a_N, a+1\bigr\}.$$

Then for any $n \geq 1$ we have

$$m \leq a_n \leq M,$$

i.e., the sequence (a_n) is bounded.

□

4.3. The arithmetic of limits

This section describes a few simple yet basic techniques that reduce the study of the convergence of a sequence to a similar study of potentially simpler sequences. Thus, we will prove that the sum of two convergent sequences is a convergent sequence etc.

Proposition 4.15 (Passage to the limit)**.** *Suppose that $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two convergent sequences,*

$$a := \lim_{n\to\infty} a_n, \;\; b = \lim_{n\to\infty} b_n.$$

The following hold.

(i) *The sequence $(a_n + b_n)_{n\geq 1}$ is convergent and*

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b.$$

(ii) *If $\lambda \in \mathbb{R}$ then*

$$\lim_{n\to\infty}(\lambda a_n) = \lambda \lim_{n\to\infty} a_n = \lambda a.$$

(iii)

$$\lim_{n\to\infty}(a_n \cdot b_n) = \bigl(\lim_{n\to\infty} a_n\bigr) \cdot \bigl(\lim_{n\to\infty} b_n\bigr) = ab.$$

(iv) *Suppose that $b \neq 0$. Then there exists $N_0 > 0$ such that $b_n \neq 0$, $\forall N > N_0$ and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

(v) *Suppose that m, M are real numbers such that $m \leq a_n \leq M$, $\forall n$. Then*

$$m \leq \lim_{n\to\infty} a_n = a \leq M.$$

Proof. (i) Because (a_n) and (b_n) are convergent, for any $\varepsilon > 0$ there exist $N_a(\varepsilon), N_b(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2}, \;\; \forall n > N_a(\varepsilon), \tag{4.9a}$$

$$|b_n - b| < \frac{\varepsilon}{2}, \;\; \forall n > N_b(\varepsilon). \tag{4.9b}$$

Let

$$N(\varepsilon) := \max\bigl\{ N_a(\varepsilon),\; N_b(\varepsilon) \bigr\}.$$

Then for any $n > N(\varepsilon)$ we have $n > N_a(\varepsilon)$ and $n > N_b(\varepsilon)$ and

$$\begin{aligned} \bigl|\, (a_n + b_n) - (a+b) \,\bigr| &= \bigl|\, (a_n - a) + (b_n - b) \,\bigr| \leq |a_n - a| + |b_n - b| \\ &\stackrel{(4.9a),(4.9b)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that $\lim_{n\to\infty}(a_n + b_n) = a + b$.

(ii) If $\lambda = 0$, then the sequence (λa_n) is the constant sequence $0, 0, 0, \dots$ and the conclusion is obvious. Assume that $\lambda \neq 0$. The sequence (a_n) is convergent so for any $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{|\lambda|}, \;\; \forall n > N(\varepsilon).$$

Hence for any $n > N(\varepsilon)$ we have

$$|\lambda a_n - \lambda a| = |\lambda| \cdot |a_n - a| < |\lambda| \cdot \frac{\varepsilon}{|\lambda|} = \varepsilon.$$

(iii) The sequences (a_n), (b_n) are convergent and thus, according to Proposition 4.14 they are bounded so that

$$\exists M > 0 : \;\; |a_n|, |b_n| \leq M, \;\; \forall n.$$

We have

$$\begin{aligned} |a_n b_n - ab| &= |(a_n b_n - a b_n) + (a b_n - ab)| \leq |a_n b_n - a b_n| + |a b_n - ab| \\ &= |b_n| \cdot |a_n - a| + |a| \cdot |b_n - b| \leq M|a_n - a| + |a| \cdot |b_n - b|. \end{aligned}$$

Part (ii) coupled with the convergence of (a_n) and (b_n) show that

$$\lim_{n\to\infty} M|a_n - a| = \lim_{n\to\infty} |a| \cdot |b_n - b| = 0.$$

Using (i) we deduce

$$\lim_{n\to\infty} \bigl(M|a_n - a| + |a| \cdot |b_n - b| \bigr) = 0.$$

The squeezing principle shows that $|a_n b_n - ab| \to 0$.

(iv) Let us first show that if $b \neq 0$, then $b_n \neq 0$ for n sufficiently large. Since $b_n \to b$ there exists $N_0 \in \mathbb{N}$ such that

$$\forall n > N_0 \;\; |b_n - b| < \frac{|b|}{2}.$$

Thus, for any $n > N_0$, we have

$$\operatorname{dist}(b_n, b) = |b_n - b| < \frac{1}{2}|b| = \frac{1}{2}\operatorname{dist}(b, 0).$$

This shows that for $n > N_0$ we cannot have $b_n = 0$. In fact

$$|b_n| > \frac{|b|}{2}, \;\; \forall n > N_0. \tag{4.10}$$

Thus, the ratio $\frac{b_n}{b_n}$ is well defined at least for $n > N_0$. We have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n| \cdot |b|}.$$

The inequality (4.10) implies

$$\frac{1}{|b_n|} < \frac{2}{|b|}, \quad \forall n > N_0.$$

Hence, for $n > N_0$ we have

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{2}{|b|^2}|b_n - b| \to 0.$$

This implies

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

Thus

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n} = \frac{a}{b}.$$

(v) We argue by contradiction. Suppose that $a > M$ or $a < m$. We discuss what happens if $a > M$, the other situation being entirely similar. Then $\delta = a - M = \operatorname{dist}(a, M) > 0$. Since $a_n \to a$, there exists $N \in \mathbb{N}$ such that if $n > N$, then

$$\operatorname{dist}(a_n, a) = |a_n - a| < \frac{\delta}{2}.$$

Thus, for $n > N_0$ we have

$$a - \frac{\delta}{2} < a_n < a + \frac{\delta}{2}.$$

Clearly $M = a - \delta < a - \frac{\delta}{2}$ and thus, a fortiori, $a_n > M$ for $n > N_0$. Contradiction! □

Corollary 4.16. *Suppose that (a_n) and (b_n) are convergent sequences such that $a_n \geq b_n$, $\forall n$. Then*

$$\lim_{n\to\infty} a_n \geq \lim_{n\to\infty} b_n.$$

Proof. Let $c_n = a_n - b_n$. Then $c_n \geq 0$ $\forall n$ and thus

$$\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n \geq 0.$$

□

Let us see how the above simple principles work in practice.

Example 4.17. We already know that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

We deduce that for any $k \in \mathbb{N}$ we have

$$\lim_{n\to\infty} \frac{1}{n^k} = 0.$$

Consider the sequence

$$a_n := \frac{5n^2 + 3n + 2}{3n^2 - 2n + 1}$$

We have

$$a_n = \frac{n^2(5+\frac{3}{n}+\frac{2}{n^2})}{n^2(3-\frac{2}{n}+\frac{1}{n^2})} = \frac{(5+\frac{3}{n}+\frac{2}{n^2})}{(3-\frac{2}{n}+\frac{1}{n^2})}.$$

Now observe that as $n \to \infty$

$$5+\frac{3}{n}+\frac{2}{n^2} \to 5,\ \ 3-\frac{2}{n}+\frac{1}{n^2} \to 3,$$

so that

$$\lim_{n\to\infty} a_n = \frac{5}{3}.$$

More generally, given $k \in \mathbb{N}$ and real numbers $a_0, b_0, \dots, a_k, b_k$ such that $b_k \neq 0$ then

$$\boxed{\lim_{n\to\infty} \frac{a_k n^k + \cdots + a_1 n + a_0}{b_k n^k + \cdots + b_1 n + b_0} = \frac{a_k}{b_k}}. \tag{4.11}$$

The proof is left to you as an exercise. □

Example 4.18. We want to show that

$$\boxed{\forall r > 1 \ \ \lim_n \frac{n}{r^n} = 0}. \tag{4.12}$$

We plan to use the Squeezing Principle and construct a sequence $(x_n)_{n\geq 1}$ of positive numbers such that

$$\frac{n}{r^n} \leq x_n \ \ \forall n \geq 2,$$

and

$$\lim_n x_n = 0.$$

Observe that since $r > 1$, we have $r - 1 > 0$. Set $a := r - 1$ so that $r = 1 + a$. Then, using Newton's binomial formula we deduce that if $n \geq 2$ then

$$r^n = (1+a)^n = 1 + \binom{n}{1}a + \binom{n}{2}a^2 + \cdots \geq 1 + \binom{n}{1}a + \binom{n}{2}a^2$$

$$= 1 + na + \frac{n(n-1)}{2}a^2 = 1 + na + \frac{a^2}{2}(n^2 - n).$$

Hence for $n \geq 2$ we have

$$\frac{1}{r^n} \leq \frac{1}{\frac{1}{2}(n^2-n)a^2 + na + 1}$$

so that

$$\frac{n}{r^n} \leq \frac{n}{\frac{a^2}{2}(n^2-n) + na + 1} =: x_n.$$

Now observe that

$$x_n = \frac{n}{n^2\big(\frac{a^2}{2}(1-\frac{1}{n}) + \frac{a}{n} + \frac{1}{n^2}\big)} = \frac{\frac{1}{n}}{\frac{a^2}{2}(1-\frac{1}{n}) + \frac{a}{n} + \frac{1}{n^2}} \xrightarrow{n\to\infty} 0.$$ □

Example 4.19. We want to show that

$$\boxed{\lim_n \sqrt[n]{n} = 1}. \tag{4.13}$$

Let $\varepsilon > 0$. The number $r_\varepsilon = 1 + \varepsilon$ is > 1. Since $\frac{n}{r_\varepsilon^n} \to 0$ we deduce that there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\frac{n}{r_\varepsilon^n} < 1, \;\; \forall n > N(\varepsilon).$$

This translates into the inequality

$$n < r_\varepsilon^n = (1+\varepsilon)^n, \;\; \forall n > N(\varepsilon).$$

In particular

$$1 \leq \sqrt[n]{n} < \sqrt[n]{(1+\varepsilon)^n} = 1 + \varepsilon.$$

We have thus proved that for any $\varepsilon > 0$ we can find $N = N(\varepsilon) \in \mathbb{N}$ so that, as soon as $n > N(\varepsilon)$ we have

$$1 \leq \sqrt[n]{n} < 1 + \varepsilon.$$

Clearly this proves the equality (4.13). □

Definition 4.20 (Infinite limits)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

(i) We say that a_n tends to ∞ as $n \to \infty$, and we write this

$$\lim_{n\to\infty} a_n = \infty$$

if

$$\forall C > 0 \;\; \exists N = N(C) \in \mathbb{N} : \;\; \forall n (n > N \Rightarrow a_n > C).$$

(ii) We say that a_n tends to $-\infty$ as $n \to \infty$, and we write this

$$\lim_{n\to\infty} a_n = -\infty$$

if

$$\forall C > 0 \;\; \exists N = N(C) \in \mathbb{N} : \;\; \forall n (n > N \Rightarrow a_n < -C).$$

□

Proposition 4.15 continues to hold if one or both of limits a, b are $\pm\infty$ provided we use the following conventions

$$\boxed{\infty + \infty = \infty \cdot \infty = \infty, \;\; \frac{C}{\infty} = 0, \;\; \forall C \in \mathbb{R}},$$

$$\boxed{C \cdot \infty = \begin{cases} \infty, & C > 0 \\ -\infty, & C < 0 \\ \text{undefined}, & C = 0, \end{cases}}$$

$$\infty - \infty = \text{undefined}, \;\; 0 \cdot \infty = \text{undefined}, \;\; \frac{\infty}{\infty} = \text{undefined}.$$

Example 4.21. (a) If we let $a_n = n$ and $b_n = \frac{1}{n}$, then Archimedes' Principle shows that $a_n \to \infty$ and $b_n \to 0$. We observe that $a_n b_n = 1 \to 1$. In this case $\infty \cdot 0 = 1$. On the other hand, if we let

$$a_n = n, \;\; b_n = \frac{1}{2^n}$$

then $a_n \to \infty$, $b_n \to 0$ and (4.12) shows that $a_n b_n \to 0$. In this case $\infty \cdot 0 = 0$.

(b) Consider the sequences $a_n = n$, $b_n = 2n$, $c_n = 3n$, $\forall n \in \mathbb{N}$. Observe that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \infty.$$

However

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\infty}{\infty} = \frac{1}{2}, \;\; \lim_{n\to\infty} \frac{a_n}{c_n} = \frac{\infty}{\infty} = \frac{1}{3}.$$

□

☛ ***Important Warning!*** When investigating limits of sequences you should keep in mind that the following arithmetic operations are treacherous and should be dealt with using *extreme care.*

$$\frac{\text{anything}}{0}, \;\; 0 \cdot \infty, \;\; \infty - \infty, \;\; \frac{\infty}{\infty}.$$

□

4.4. Convergence of monotone sequences

The definition of convergence has one drawback: to verify that a sequence is convergent using the definition we need to a priori know its limit. In most cases this is a nearly impossible job. In this and the next section we will discuss techniques for proving the convergence of a sequence without knowing the precise value of its limit.

Theorem 4.22 (Weierstrass)**.** *Any bounded and monotone sequence is convergent.*

Proof. Suppose that (a_n) is a bounded and monotone sequence, i.e., it is either non-decreasing, or non-increasing. We investigate only the case when (a_n) is nondecreasing, i.e.,

$$a_1 \leq a_2 \leq a_3 \leq \cdots .$$

The situation when (a_n) is non-increasing is completely similar.

The set of real numbers

$$A := \{ a_n; \;\; n \geq 1 \}$$

is bounded because the sequence (a_n) is bounded. The Completeness Axiom implies it has a least upper bound

$$a := \sup A.$$

We will prove that

$$\lim_{n\to\infty} a_n = a. \tag{4.14}$$

Since a is an upper bound for the sequence we have

$$a_n \leq a, \;\; \forall n. \tag{4.15}$$

Proposition 2.21 implies that for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$a - \varepsilon < a_{N(\varepsilon)}.$$

Since (a_n) is nondecreasing we deduce that

$$a - \varepsilon < a_{N(\varepsilon)} \le a_n, \;\; \forall n > N(\varepsilon) \tag{4.16}$$

Putting together (4.15) and (4.16) we deduce that

$$\forall \varepsilon > 0 \;\; \exists N = N(\varepsilon) \in \mathbb{N} : \;\; \forall n \; (n > N(\varepsilon) \Rightarrow a - \varepsilon < a_n \le a).$$

This implies the claimed convergence (4.14) because $a - \varepsilon < a_n \le a \Rightarrow |a_n - a| < \varepsilon$. □

We will spend the rest of this section presenting applications of the above *very important* theorem.

Example 4.23 (L. Euler)**.** Consider the sequence of positive numbers

$$x_n = \left(1 + \frac{1}{n}\right)^n, \;\; n \in \mathbb{N}.$$

We will prove that this sequence is convergent. Its limit is called *Euler's number e*.

We plan to use Weierstrass' theorem applied to a new sequence of positive numbers

$$y_n = \left(1 + \frac{1}{n}\right)^{n+1}, \;\; n \in \mathbb{N}.$$

Note that

$$y_n = \left(\frac{n+1}{n}\right)^{n+1}$$

and for $n \ge 2$ we have

$$\begin{aligned} \frac{y_{n-1}}{y_n} &= \frac{\left(\frac{n}{n-1}\right)^n}{\left(\frac{n+1}{n}\right)^{n+1}} = \left(\frac{n}{n-1}\right)^n \cdot \left(\frac{n}{n+1}\right)^{n+1} \\ &= \frac{n^{2n+1}}{(n-1)^n (n+1)^n \cdot (n+1)} = \frac{n^{2n}}{(n^2-1)^n} \cdot \frac{n}{n+1} \\ &= \left(\frac{n^2}{n^2-1}\right)^n \cdot \frac{n}{n+1} = \underbrace{\left(1 + \frac{1}{n^2-1}\right)^n}_{=:q_n} \cdot \frac{n}{n+1}. \end{aligned}$$

Bernoulli's inequality implies that

$$q_n := \left(1 + \frac{1}{n^2-1}\right)^n \ge 1 + \frac{n}{n^2-1} > 1 + \frac{n}{n^2} = 1 + \frac{1}{n} = \frac{n+1}{n}.$$

Hence

$$\frac{y_{n-1}}{y_n} = q_n \cdot \frac{n}{n+1} > \frac{n+1}{n} \cdot \frac{n}{n+1} = 1.$$

Hence $y_{n-1} > y_n \; \forall n \ge 2$, i.e., the sequence (y_n) is decreasing. Since it is bounded below by 1 we deduce that the sequence (y_n) is convergent.

Now observe that $y_n = x_n \cdot \left(1 + \frac{1}{n}\right) = x_n \cdot \frac{n+1}{n}$ so that

$$x_n = y_n \cdot \frac{n}{n+1}.$$

Since

$$\lim_n \frac{n}{n+1} = 1$$

we deduce that (x_n) is convergent and has the same limit as the sequence (y_n). □

Definition 4.24. The *Euler number*, denoted e is defined to be

$$\boxed{e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}. \qquad \square$$

The arguments in Example 4.23 show that that

$$4 = y_1 \geq e \geq 2.$$

Using more sophisticated methods one can show that

$$e = 2.71828182845905\dots .$$

Example 4.25 (Babylonians and I. Newton)**.** Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined recursively by the requirements

$$x_1 = 1,\;\; x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right),\;\; \forall n \in \mathbb{N}.$$

Thus

$$x_2 = \frac{1}{2}\left(1+\frac{2}{1}\right) = \frac{3}{2},$$

$$x_3 = \frac{1}{2}\left(\frac{3}{2}+\frac{2}{\frac{3}{2}}\right) = \frac{1}{2}\left(\frac{3}{2}+\frac{4}{3}\right) = \frac{17}{12} \text{ etc.}$$

We want to prove that this sequence converges to $\sqrt{2}$. We proceed gradually.

Lemma 4.26.

$$x_n \geq \sqrt{2},\;\; \forall n \geq 2. \tag{4.17}$$

Proof. Multiplying with $2x_n$ both sides of the equality

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

we deduce $2x_nx_{n+1} = x_n^2 + 2$, or equivalently

$$x_n^2 - 2x_{n+1}x_n + 2 = 0. \tag{4.18}$$

This shows that the quadratic equation

$$t^2 - 2x_{n+1}t + 2 = 0$$

has at least one real solution, $t = x_{n+1}$ so that (see Exercise 3.10)

$$\Delta = 4x_{n+1}^2 - 8 \geq 0,$$

i.e., $x_{n+1}^2 \geq 2$, or $x_{n+1} \geq \sqrt{2}$, $\forall n \in \mathbb{N}$. □

Lemma 4.27. *For any $n \geq 2$ we have*

$$x_{n+1} \leq x_n.$$

Proof. Let $n \geq 2$. We have

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{1}{2}\frac{x_n^2 - 2}{x_n} \overset{(4.17)}{\geq} 0.$$

$\square$

Thus the sequence $(x_n)_{n \geq 2}$ is decreasing and bounded below and thus it is convergent. Denote by $\bar{x}$ the limit. The inequality (4.17) implies that $\bar{x} \geq \sqrt{2}$. Letting $n \to \infty$ in (4.18) we deduce

$$\bar{x}^2 - 2\bar{x}^2 + 2 = 0 \Rightarrow 2 = \bar{x}^2 \Rightarrow \bar{x} = \sqrt{2}.$$

For example

$$x_2 = 1.5, \quad x_3 = 1.4166..., \quad x_4 := 1.4142..., \quad x_5 := 1.4142....$$

Note that

$$(1.4142)^2 = 1.99996164.$$

$\square$

Theorem 4.28 (Nested Intervals Theorem)**.** *Consider a* nested sequence *of* ***closed*** *intervals* $[a_n, b_n]$, $n \in \mathbb{N}$, *i.e.,*

$$[a_1, b_1] \supset [a_2, b_2] \supset [a_3, b_3] \supset \cdots .$$

Then there exists $x \in \mathbb{R}$ *that belongs to all the intervals, i.e.,*

$$\bigcap_{n \in \mathbb{N}} [a_n, b_n] \neq \emptyset.$$

Proof. The nesting condition implies that for any $n \in \mathbb{N}$ we have

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n.$$

This shows that the sequence (a_n) is non-decreasing and bounded while the sequence (b_n) is non-increasing. Therefore, these sequences are convergent and we set

$$a := \lim_n a_n, \quad b := \lim_n b_n$$

the condition $a_n \leq b_n$, $\forall n$ implies that

$$a_n \leq a \leq b \leq b_n, \quad \forall n.$$

Hence $[a, b] \subset [a_n, b_n]$, $\forall n$. $\square$

Theorem 4.29 (Bolzano-Weierstrass)**.** *Any bounded sequence has a convergent subsequence.*

Proof. Let (x_n) be a bounded sequence of real numbers. Thus, there exist real numbers a_1, b_1 such that $x_n \in [a_1, b_1]$, for all n. We set

$$n_1 := 1.$$

Divide the interval $[a_1, b_1]$ into two intervals of equal length. At least one of these intervals will contain infinitely many terms of the sequence (x_n). Pick such an interval and denote it by $[a_2, b_2]$. Thus

$$[a_1, b_1] \supset [a_2, b_2], \quad b_2 - a_2 = \frac{1}{2}(b_1 - a_1).$$

Choose $n_2 > 1$ such that $x_{n_2} \in [a_2, b_2]$. We now proceed inductively.

Suppose that we have produced the intervals

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_k, b_k]$$

and the natural numbers $n_1 < n_2 < \cdots < n_k$ such that

$$b_2 - a_2 = \frac{1}{2}(b_1 - a_1), \;\; b_3 - a_3 = \frac{1}{2}(b_2 - a_2), \;\; b_k - a_k = \frac{1}{2}(b_{k-1} - a_{k-1}),$$

$$x_{n_1} \in [a_1, b_1], \;\; x_{n_2} \in [a_2, b_2], \cdots x_{n_k} \in [a_k, b_k],$$

and the interval $[a_k, b_k]$ contains infinitely many terms of the sequence (x_n). We then divide $[a_k, b_k]$ into two intervals of equal lengths,. One of them will contain infinitely many terms of (x_n). Denote that interval by $[a_{k+1}, b_{k+1}]$. We can then find a a natural number $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in [a_{k+1}, b_{k+1}]$. By construction

$$b_{k+1} - a_{k+1} = \frac{1}{2}(b_k - a_k) = \cdots = \frac{1}{2^k}(a_1 - b_1).$$

We have thus produced sequences (a_k), (b_k) (x_{n_k}) with the properties

$$a_1 \leq a_2 \leq \cdots \leq a_k \leq x_{n_k} \leq b_k \leq \cdots \leq b_2 \leq b_1, \tag{4.19a}$$

$$b_k - a_k = \frac{1}{2^{k-1}}(b_1 - a_1). \tag{4.19b}$$

The inequalities (4.19a) show that the sequences (a_k) and (b_k) are monotone and bounded, and thus have limits which we denote by a and b respectively. By letting $k \to \infty$ in (4.19b) we deduce that $a = b$.

The subsequence (x_{n_k}) is squeezed between two sequences converging to the same limit so the squeezing theorem implies that it is convergent. □

Definition 4.30. A *limit point* of a sequence of real numbers (x_n) is a real number which is the limit of some subsequence of the original sequence (x_n). □

Example 4.31. Consider the sequence

$$x_n = (-1)^n + \frac{1}{n}, \;\; n \in \mathbb{N}.$$

Thus

$$x_{2n} = 1 + \frac{1}{2n}, \;\; x_{2n+1} = -1 + \frac{1}{2n+1}.$$

Then the numbers 1 and -1 are limit points of this sequence because

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty}\left(1 + \frac{1}{2n}\right) = 1,$$

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty}\left(-1 + \frac{1}{2n+1}\right) = -1.$$ □

4.5. Fundamental sequences and Cauchy's characterization of convergence

We know that any convergent sequence is bounded, so that boundedness is a necessary condition for a sequence to be convergent. However, it is not also a sufficient condition. For example, the sequence

$$1, -1, 1, -1, \dots,$$

is bounded, but it is not convergent.

Weierstrass's theorem on bounded monotone sequences shows that monotonicity is a sufficient condition for a bounded sequence to be convergent. However, monotonicity is not a necessary condition for convergence. Indeed, the sequence

$$x_n = \frac{(-1)^n}{n}, \quad n \in \mathbb{N}$$

converges to zero, yet it is not monotone because the even order terms are positive while the odd order terms are negative. In this subsection we will present a fundamental necessary and sufficient condition for a sequence to be convergent that makes no reference to the precise value of the limit. We begin by defining a very important concept.

Definition 4.32. A sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ is called *fundamental* (or *Cauchy*) if the following holds:

$$\forall \varepsilon > 0 \;\; \exists N = N(\varepsilon) \in \mathbb{N} \;\text{ such that }\; \forall m, n > N(\varepsilon): \;\; |a_m - a_n| < \varepsilon. \tag{4.20}$$

□

Theorem 4.33 (Cauchy)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then the following statements are equivalent.*

(i) *The sequence (a_n) is convergent.*

(ii) *The sequence (a_n) is fundamental.*

Proof. (i) ⇒ (ii). We know that there exists $a \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \;\; \exists N = N(\varepsilon) \in \mathbb{N}: \;\; \forall n > N(\varepsilon) \;\; |a_n - a| < \varepsilon.$$

Observe that for any $m, n > N(\varepsilon/2)$ we have

$$|a_m - a_n| \leq |a_m - a| + |a - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves that (a_n) is fundamental.

(ii) ⇒ (i) This is the "meatier" part of the theorem. We will reach the conclusion in three conceptually distinct steps.

1. Using the fact that the sequence (a_n) is fundamental we will prove that it is bounded.
2. Since (a_n) is bounded, the Bolzano-Weierstrass theorem implies that it has a subsequence that converges to a real number a.
3. Using the fact that the sequence (a_n) is fundamental we will prove that it converges to the real number a found above.

Here are the details. Since (a_n) is fundamental, there exists $n_1 > 0$ such that, for any $m, n \geq n_1$ we have $|a_m - a_n| < 1$. Hence if we let $m = n_1$ we deduce that for any $n \geq n_1$ we have

$$|a_{n_1} - a_n| < 1 \Rightarrow a_{n_1} - 1 < a_n < a_{n_1} + 1, \ \ \forall n \geq n_1.$$

Now let

$$m := \min\{a_1, a_2, \dots, a_{n_1-1}, a_{n_1} - 1\}, \ \ M := \max\{a_1, a_2, \dots, a_{n_1-1}, a_{n_1} + 1\}.$$

Clearly

$$m \leq a_n \leq M, \ \ \forall n \in \mathbb{N}$$

so that the sequence (a_n) is bounded.

Invoking the Bolzano-Weierstrass theorem we deduce that there exists a subsequence $(a_{n_k})_{k\geq 1}$ and a real number a such that

$$\lim_{k\to\infty} a_{n_k} = a.$$

Let $\varepsilon > 0$. Since $a_{n_k} \to a$ as $k \to \infty$ we deduce that

$$\exists K = K(\varepsilon) \in \mathbb{N} \ \text{ such that } \ \forall k > K(\varepsilon) : \ \ |a_{n_k} - a| < \frac{\varepsilon}{2}.$$

On the other hand, the sequence $(a_n)_{n\in\mathbb{N}}$ is fundamental so that

$$\exists N' = N'(\varepsilon) \in \mathbb{N} \ \text{ such that } \ \forall m, n > N'(\varepsilon) : \ \ |a_m - a_n| < \frac{\varepsilon}{2}.$$

Now choose a natural number $k_0(\varepsilon) > K(\varepsilon)$ such that $n_{k_0(\varepsilon)} > N'(\varepsilon)$. Define

$$N(\varepsilon) = n_{k_0(\varepsilon)}.$$

If $n > N(\varepsilon)$ then $n, n_{k_0} > N'(\varepsilon)$ and thus

$$|a_n - a_{n_{k_0}}| < \frac{\varepsilon}{2}.$$

On the other hand, since $k_0(\varepsilon) > K(\varepsilon)$ we deduce that

$$|a_{n_{k_0}} - a| < \frac{\varepsilon}{2}.$$

Hence, for any $n > N(\varepsilon)$ we have

$$|a_n - a| \leq |a_n - a_{n_{k_0}}| + |a_{n_{k_0}} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

Since ε was arbitrary we conclude that (a_n) converges to a. □

4.6. Series

Often one has to deal with sums of infinitely many terms. Such a sum is called a *series*. Here is the precise definition.

Definition 4.34. The *series* associated to a sequence $(a_n)_{n\geq 0}$ of real numbers is the ***new*** sequence $(s_n)_{n\geq 0}$ defined by the *partial sums*

$$s_0 = a_0, \ \ s_1 = a_0 + a_1, \ \ s_2 = a_0 + a_1 + a_2, \dots, s_n = a_0 + a_1 + \cdots + a_n = \sum_{i=0}^{n} a_i, \ \dots .$$

The series associated to the sequence $(a_n)_{n\geq 0}$ is denoted by the symbol

$$\sum_{n=0}^{\infty} a_n \quad \text{or} \quad \sum_{n\geq 0} a_n.$$

The series is called *convergent* if the sequence of partial sums $(s_n)_{n\geq 0}$ is convergent. The limit $\lim_{n\to\infty} s_n$ is called *the sum* series. We will use the notation

$$\sum_{n\geq 0} a_n = S$$

to indicate that the series is convergent and its sum is the real number S. □

Example 4.35 (Geometric series. Part 1)**.** Let $r \in (-1,1)$. The *geometric series*

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots$$

is convergent and we have the following *very useful equality*

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}}. \tag{4.21}$$

Indeed, the n-th partial sum of this series is

$$s_n = 1 + r + \cdots + r^n = \frac{1-r^{n+1}}{1-r}.$$

Example 4.12 shows that when $|r| < 1$ we have $\lim_n r^{n+1} = 0$ so that

$$\sum_{n=0}^{\infty} r^n = \lim_n s_n = \frac{1}{1-r}.$$

Observe that if we set $r = \frac{1}{2}$ in (4.21) we deduce

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

□

The proof of the following result is left to you as an exercise.

Proposition 4.36. *Consider two series*

$$\sum_{n\geq 0} a_n \quad \textit{and} \quad \sum_{n\geq 0} a'_n$$

such that there exists $N_0 > 0$ with the property

$$a_n = a'_n \quad \forall n > N_0.$$

Then

$$\sum_{n\geq 0} a_n \ \textit{is convergent} \Longleftrightarrow \sum_{n\geq 0} a'_n \ \textit{is convergent.}$$

□

Proposition 4.37. *If the series $\sum_{n=0}^{\infty} a_n$ is convergent, then*

$$\lim_{n\to\infty} a_n = 0.$$

Proof. Observe that for $n \geq 1$

$$s_n = a_0 + a_1 + \cdots + a_{n-1} + a_n = s_{n-1} + a_n.$$

Hence

$$a_n = s_n - s_{n-1}.$$

The sequences $(s_n)_{n\geq 1}$ and $(s_{n-1})_{n\geq 1}$ converge to the same finite limit so that

$$\lim_n a_n = \lim_n s_n - \lim_n s_{n-1} = 0.$$

□

Example 4.38 (Geometric series. Part 2)**.** Let $|r| \geq 1$. Then the geometric series

$$\sum_{n=0}^{\infty} r^n$$

is divergent. Indeed, if it were convergent, then the above proposition would imply that $r^n \to 0$ as $n \to \infty$. This is not the case when $|r| \geq 1$. □

Proposition 4.39. *A series of positive numbers*

$$\sum_{n\geq 0} a_n, \quad a_n > 0 \;\; \forall n$$

is convergent if and only if the sequence of partial sums

$$s_n = a_0 + \cdots + a_n$$

is bounded.

Proof. Observe that the sequence of partial sums is increasing since

$$s_{n+1} - s_n = a_{n+1} > 0, \;\; \forall n.$$

If the sequence (s_n) is also bounded, then Weierstrass' Theorem on monotone sequences implies that it must be convergent.

Conversely, if the sequence (s_n) is convergent, then Proposition 4.14 shows that it must also be bounded. □

Example 4.40. (a) Consider the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

This is a series with positive terms. Observe that

$$s_1 = 1 \geq 1, \;\; s_2 = 1 + \frac{1}{2} \geq 1 + \frac{1}{2},$$

$$s_{2^2} = s_4 = s_2 + \frac{1}{3} + \frac{1}{4} > s_2 + \frac{1}{4} + \frac{1}{4} = s_2 + \frac{1}{2} = 1 + \frac{2}{2}$$

$$s_{2^3} = s_8 = s_4 + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\text{4 terms}} > 1 + \frac{2}{2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\text{4 terms}}$$
$$> 1 + \frac{2}{2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\text{4 terms}} = 1 + \frac{3}{2}.$$

Thus

$$s_{2^3} > 1 + \frac{3}{2}.$$

We want to prove that

$$s_{2^n} > 1 + \frac{n}{2}, \quad \forall n \geq 2. \tag{4.22}$$

We have shown this for $n = 2$ and $n = 3$. The general case follows inductively. Observe that $2^{n+1} = 2 \cdot 2^n = 2^n + 2^n$ and thus

$$s_{2^{n+1}} = s_{2^n} + \underbrace{\frac{1}{2^n + 1} + \cdots + \frac{1}{2^{n+1}}}_{2^n\text{-terms}}$$
$$> s_{2^n} + \underbrace{\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}}}_{2^n\text{-terms}} = s_{2^n} + \frac{2^n}{2^{n+1}} = s_{2^n} + \frac{1}{2}$$

(use the inductive assumption)

$$> 1 + \frac{n}{2} + \frac{1}{2}.$$

This proves that (4.22) which shows that the sequence s_{2^n} is not bounded. Invoking Proposition 4.39 we conclude that the harmonic series is not convergent.

(b) Let $r > 1$ be a rational number and consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^r}.$$

We have

$$s_2 = 1 + \frac{1}{2^r},$$
$$s_4 = s_2 + \frac{1}{3^r} + \frac{1}{4^r} < s_2 + \frac{1}{2^r} + \frac{1}{2^r} < s_2 + \frac{2}{2^r} = \frac{1}{2^r} + 1 + \frac{1}{2^{(r-1)}},$$
$$s_{2^3} = s_8 = s_4 + \frac{1}{5^r} + \frac{1}{6^r} + \frac{1}{7^r} + \frac{1}{8^r} < s_4 + \frac{4}{4^r} = \frac{1}{2^r} + 1 + \frac{1}{2^{(r-1)}} + \frac{1}{2^{2(r-1)}}.$$

We claim that for any $n \geq 1$ we have

$$s_{2^{n+1}} < \frac{1}{2^r} + 1 + \frac{1}{2^{(r-1)}} + \frac{1}{2^{2(r-1)}} + \cdots + \frac{1}{2^{n(r-1)}} \tag{4.23}$$

We argue inductively. The result is clearly true for $n = 1, 2$. We assume it is true for n and we prove it is true for $n + 1$. We have

$$s_{2^{n+1}} = s_{2^n} + \underbrace{\frac{1}{(2^n + 1)^r} + \frac{1}{(2^n + 2)^r} + \cdots + \frac{1}{(2^{n+1})^r}}_{2^n \text{ terms}}$$

$$< s_{2^n} + \underbrace{\frac{1}{(2^n)^r} + \frac{1}{(2^n)^r} + \cdots + \frac{1}{(2^n)^r}}_{2^n \text{ terms}} = s_{2^n} + \frac{1}{2^{n(r-1)}}$$

(use the induction assumption)

$$< \frac{1}{2^r} + 1 + \frac{1}{2^{(r-1)}} + \frac{1}{2^{2(r-1)}} + \cdots + \frac{1}{2^{(n-1)(r-1)}} + \frac{1}{2^{n(r-1)}}.$$

If we set

$$q := \frac{1}{2^{r-1}} = \left(\frac{1}{2}\right)^{r-1},$$

then we observe that the condition $r > 1$ implies $q \in (0, 1)$ and we can rewrite (4.23) as

$$s_{2^{n+1}} < \frac{1}{2^r} + 1 + q + \cdots + q^n < \frac{1}{2^r} + \frac{1}{1-q}, \quad \forall n \in \mathbb{N}.$$

This implies that the sequence (s_{2^n}) is bounded and thus the series

$$\sum_{n=1}^{\infty} \frac{1}{n^r}$$

is convergent. Its sum is denoted by $\zeta(r)$ and it is called *Riemann zeta function* For most r's, the actual value $\zeta(r)$ is not known. However, L. Euler has computed the values $\zeta(r)$ when r is an even natural number. For example

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

All the known proofs of the above equality are very ingenious. □

Given a series $\sum_{n=0}^{\infty} a_n$ and natural numbers $m < n$ we have

$$s_n - s_m = (a_{m+1} + a_{m+1} + \cdots + a_n) = \sum_{k=m+1}^{n} a_k.$$

Cauchy's Theorem 4.33 implies the following useful result.

Theorem 4.41 (Cauchy)**.** *Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers. Then the following statements are equivalent.*

(i) *The series $\sum_{n=0}^{\infty} a_n$ is convergent.*

(ii)

$$\forall \varepsilon > 0 \;\; \exists N = N(\varepsilon) \in \mathbb{N} \;\; \textit{such that} \;\; \forall n > m > N(\varepsilon) \;\; |a_{m+1} + \cdots + a_n| < \varepsilon.$$

□

Theorem 4.42 (Comparison principle)**.** *Suppose that*

$$\sum_{n \geq 0} a_n \;\; \textit{and} \;\; \sum_{n \geq 0} b_n$$

are two series of positive real numbers such that

$$\exists N_0 \in \mathbb{N} \;\; \textit{such that} \;\; \forall n > N_0 : \;\; a_n < b_n.$$

Then the following hold.

(a) $\sum_{n\geq 0} a_n$ *divergent* $\Rightarrow \sum_{n\geq 0} b_n$ *divergent.*

(b) $\sum_{n\geq 0} b_n$ *convergent* $\Rightarrow \sum_{n\geq 0} a_n$ *convergent.*

Proof. We set

$$s_n(a) = \sum_{k=0}^{n} a_n, \quad s_n(b) = \sum_{k=1}^{n} b_n.$$

In view of Proposition 4.36 the convergence or divergence of a series is not affected if we modify finitely many of its terms. Thus, we may assume that

$$a_n \leq b_n, \quad \forall n \geq 0.$$

In particular, we have

$$s_n(a) \leq s_n(b), \quad \forall n \geq 0. \tag{4.24}$$

Note that since the terms a_n are *positive*

$$\sum_{n\geq 0} a_n \ \text{ divergent} \Rightarrow s_n(a) \to \infty \Rightarrow s_n(b) \to \infty \Rightarrow \sum_{n\geq 0} b_n \ \text{ divergent}$$

and

$$\sum_{n\geq 0} b_n \ \text{ convergent} \Rightarrow s_n(b) \ \text{ bounded} \Rightarrow s_n(a) \ \text{ bounded} \Rightarrow \sum_{n\geq 0} a_n \ \text{ convergent.}$$

□

The above result has an immediate and very useful consequence whose proof is left to you as an exercise.

Corollary 4.43. *Suppose that*

$$\sum_{n\geq 0} a_n \ \textit{and} \ \sum_{n\geq 0} b_n$$

are two series with positive terms.

(a) If the sequence $(\frac{a_n}{b_n})_{n\geq 0}$ *is convergent and the series* $\sum_{n\geq 0} b_n$ *is convergent, then the series* $\sum_{n\geq 0} a_n$ *is also convergent.*

(b) If the sequence $(\frac{a_n}{b_n})_{n\geq 0}$ *has a limit* r *which is either positive,* $r > 0$*, or* $r = \infty$ *and the series* $\sum_{n\geq 0} b_n$ *is divergent, then the series* $\sum_{n\geq 0} a_n$ *is also divergent.* □

Example 4.44 (L. Euler)**.** Consider the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots . \tag{4.25}$$

Observe that if $n \geq 2$, then

$$\frac{1}{n!} = \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n} \leq \underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{(n-1)-\text{times}} = \frac{1}{2^{n-1}} = \frac{2}{2^n}.$$

Since the series

$$\sum_{n\geq 0}\frac{2}{2^n}$$

is convergent we deduce from the Comparison Principle that the series (4.25) is also convergent. Its sum is the Euler number

$$\sum_{n=0}^{\infty}\frac{1}{n!}=e=\lim_n\left(1+\frac{1}{n}\right)^n. \tag{4.26}$$

This is a nontrivial result. We will describe a more conceptual proof in Corollary 8.8. However, that proof relies on the full strength of differential calculus.

Here is an elementary proof. We set

$$e_n:=\left(1+\frac{1}{n}\right)^n,\;\; s_n=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!},\;\;\forall n\in\mathbb{N},$$

We will prove two things.

$$e_n<s_n,\;\;\forall n\geq 1 \tag{4.27a}$$

$$s_k\leq e,\;\;\forall k\geq 1. \tag{4.27b}$$

Assuming the validity of the above inequalities, we observe that by letting $n\to\infty$ in (4.27a) we deduce that

$$e\leq\lim_n s_n.$$

On the other hand, if we let $k\to\infty$ in (4.27b), then we conclude that

$$\lim_k s_k\leq e.$$

Hence (4.27a, 4.27b) imply that

$$e=\lim_n s_n=\sum_{n=0}^{\infty}\frac{1}{n!}.$$

Proof of (4.27a). Using Newton's binomial formula we deduce

$$\begin{aligned}
e_n&=\left(1+\frac{1}{n}\right)^n=1+\binom{n}{1}\frac{1}{n}+\binom{n}{2}\frac{1}{n^2}+\cdots+\binom{n}{n}\frac{1}{n^n}\\
&=1+\frac{n}{1!}\frac{1}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}+\cdots+\frac{n(n-1)\cdots 1}{n!}\frac{1}{n^n}\\
&=1+\frac{n}{n}\frac{1}{1!}+\underbrace{\frac{n(n-1)}{n^2}}_{<1}\cdot\frac{1}{2!}+\underbrace{\frac{n(n-1)(n-2)}{n^3}}_{<1}\cdot\frac{1}{3!}+\cdots+\underbrace{\frac{n(n-1)\cdots 1}{n^n}}_{<1}\cdot\frac{1}{n!}\\
&<1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}=s_n.
\end{aligned}$$

Proof of (4.27b). Fix $k\in\mathbb{N}$. Then from the same formula above we deduce that if $k\leq n$, then

$$e_n=1+\frac{n}{1!}\frac{1}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}+\cdots+\frac{n(n-1)\cdots 1}{n!}\frac{1}{n^n}$$

(neglect the terms containing the powers $\frac{1}{n^j}$, $j>k$)

$$\begin{aligned}
&>1+\frac{n}{1!}\frac{1}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}+\cdots+\frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k}\\
&=1+\frac{1}{1!}+\frac{n-1}{n}\frac{1}{2!}+\frac{n-1}{n}\frac{n-2}{n}\cdot\frac{1}{3!}+\cdots+\frac{n-1}{n}\cdots\frac{n-k+1}{n}\cdot\frac{1}{k!}\\
&=1+\frac{1}{1!}+\left(1-\frac{1}{n}\right)\frac{1}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{3!}+\cdots+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{1}{k!}.
\end{aligned}$$

If we let $n\to\infty$, while keeping k fixed we deduce

$$\begin{aligned}
e&=\lim_{n\to\infty}e_n\\
&\geq 1+\frac{1}{1!}+\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\frac{1}{2!}+\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{1}{3!}+\cdots
\end{aligned}$$

$$\cdots + \lim_{n\to\infty}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{1}{k!} = s_k.$$

Let us now estimate the error

$$\varepsilon_n = e - s_n = \left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots\right) - \left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)$$
$$= \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots.$$

Clearly $\varepsilon_n > 0$ and

$$\varepsilon_n = \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\cdots\right)$$
$$< \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)^2}+\frac{1}{(n+2)^3}+\cdots\right)$$
$$= \frac{1}{(n+1)!}\cdot\frac{1}{1-\frac{1}{n+2}} = \frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}.$$

For example, if we let $n = 6$, then we deduce that

$$0 < \varepsilon_6 < \frac{8}{7\cdot 6!} = \frac{8}{7\cdot 720} \approx 0.0002\ldots.$$

This shows that s_5 computes e with a 2-decimal precision. We have

$$s_5 = 1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120} = 2.71\ldots,$$

so that

$$e = 2.71\ldots$$ □

Definition 4.45. The series of real numbers

$$\sum_{n\geq 0} a_n$$

is called *absolutely convergent* if the series of absolute values

$$\sum_{n\geq 0} |a_n|$$

is convergent. □

Theorem 4.46. *If the series*

$$\sum_{n\geq 0} a_n$$

is absolutely convergent, then it is also convergent.

Proof. Since

$$\sum_{n\geq 0} |a_n|$$

is convergent, then Theorem 4.41 implies that

$$\forall \varepsilon > 0 \;\; \exists N = N(\varepsilon) \in \mathbb{N}: \;\; \forall n > m > N(\varepsilon): \;\; |a_{m+1}| + \cdots + |a_n| < \varepsilon.$$

On the other hand, we observe that

$$|a_{m+1}+\cdots+a_n| \leq |a_{m+1}| + \cdots + |a_n|$$

so that

$$\forall \varepsilon > 0 \;\; \exists N = N(\varepsilon) \in \mathbb{N}: \;\; \forall n > m > N(\varepsilon): \;\; |a_{m+1}+\cdots+a_n| < \varepsilon.$$

Invoking Theorem 4.41 again we deduce that the series $\sum_{n\geq 0} a_n$ is convergent as well. □

The Comparison Principle has the following immediate consequence.

Corollary 4.47 (Weierstrass M-test)**.** *Consider two series*

$$\sum_{n\geq 0} a_n,\ \ \sum_{n\geq 0} b_n$$

such that $b_n > 0$ *for any* n *and there exists* $N_0 \in \mathbb{N}$ *such that*

$$|a_n| < b_n,\ \ \forall n > N_0.$$

Then the series $\sum_{n\geq 0} a_n$ *converges absolutely.* □

The Weierstrass M-test leads to a simple but very useful convergence test, called *the D'Alembert test* or the *ratio test.*

Corollary 4.48 (Ratio Test)**.** *Let*

$$\sum_{n\geq 0} a_n$$

be a series such that $a_n \neq 0\ \forall n$ *and the limit*

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \geq 0$$

exists, but it could also be infinite. Then the following hold.

(i) *If* $L < 1$, *then the series* $\sum_{n\geq 0} a_n$ *is absolutely convergent.*

(ii) *If* $L > 1$ *then the series* $\sum_{n\geq 0} a_n$ *is not convergent.*

Proof. (i) We know that $L < 1$. Choose r such that $L < r < 1$. Since

$$\frac{|a_{n+1}|}{|a_n|} \to L$$

there exists $N_0 \in \mathbb{N}$ such that

$$\frac{|a_{n+1}|}{|a_n|} \leq r,\ \ \forall n > N_0 \iff |a_{n+1}| \leq |a_n| r,\ \ \forall n > N_0.$$

We deduce that

$$|a_{N_0+1}| \leq |a_{N_0}| r,\ \ |a_{N_0+2}| \leq |a_{N_0+1}| r \leq |a_{N_0}| r^2,$$

and, inductively

$$|a_{N_0+k}| \leq r^k |a_{N_0}|,\ \ \forall k \in \mathbb{N}.$$

If we set $n = N_0 + k$ so that $k = n - N_0$, then we conclude from above that for any, $n > N_0$ we have

$$|a_n| \leq |a_{N_0}| t^{n-N_0} = \underbrace{\frac{|a_{N_0}|}{r^{N_0}}}_{=:C} r^n.$$

In other words

$$|a_n| \leq C r^n,\ \ \forall n \geq N_0.$$

The series geometric series $\sum_{n\geq 0} b_n$, $b_n = Cr^n$, is convergent for $r \in (0,1)$ and we deduce from Weierstrass' Test that the series $\sum_{n\geq 0} |a_n|$ is also convergent.

(ii) We argue by contradiction and assume that the series $\sum_{n\geq 0}|a_n|$ is convergent. Since $L>1$ we deduce that there exists a $N_0\in\mathbb{N}$ such that

$$\frac{|a_{n+1}|}{|a_n|}>1,\ \ \forall n>N_0 \iff |a_{n+1}|>|a_n|,\ \ \forall n>N_0.$$

Since the series $\sum_{n\geq 0}$ we deduce that $\lim_n a_n=0$. On the other hand, $|a_n|>|a_{N_0}|$ for $n>N_0$so that

$$0=\lim_n |a_n|\geq |a_{N_0}|>0.$$

This contradiction shows that the series $\sum_{n\geq 0}|a_n|$ cannot be convergent. □

Example 4.49. (a) Consider the series

$$\sum_{n\geq 1}(-1)^n\frac{n^2}{2^n}.$$

Then

$$\frac{|a_{n+1}|}{|a_n|}=\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}=\frac{1}{2}\left(\frac{n+1}{n}\right)^2\to\frac{1}{2}\to\frac{1}{2}\ \text{ as }\ n\to\infty.$$

The D'Alembert test implies that the series is absolutely convergent.

(b) Consider the series

$$\sum_{n\geq 1}\frac{1}{\sqrt{n(n+1)}}.$$

We observe that

$$\frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}}=\frac{n}{\sqrt{n(n+1)}}=\frac{n}{\sqrt{n^2(1+\frac{1}{n})}}=\frac{1}{\sqrt{1+\frac{1}{n}}}.$$

Hence

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}}=1$$

so that there exists $N_0>0$ such that

$$\frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}}>\frac{1}{2}\ \ \forall n>N_0,$$

i.e.,

$$\frac{1}{\sqrt{n(n+1)}}>\frac{1}{2n},\ \ \forall n>N_0.$$

In Example 4.40(a) we have shown that the series $\sum_{n\geq 1}\frac{1}{2n}$ is divergent. Invoking the comparison principle we deduce that the series $\sum_{n\geq 1}\frac{1}{\sqrt{n(n+1)}}$ is also divergent. □

Definition 4.50. A series is called *conditionally convergent* if it is convergent, but not absolutely convergent. □

Example 4.51. Consider the series

$$\sum_{n\geq 0}\frac{(-1)^n}{n+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots.$$

Example 4.40(a) shows that this series is not absolutely convergent. However, it is a convergent series. To see this observe first that

$$s_0=1,\ \ s_2=s_0-\frac{1}{2}+\frac{1}{3}=s_0-\left(\frac{1}{2}-\frac{1}{3}\right)<s_0,$$

$$s_{2n+2}=s_{2n}-\frac{1}{(2n+2)}+\frac{1}{2n+3}=s_{2n}-\left(\frac{1}{2n+2}-\frac{1}{2n+3}\right)<s_{2n}.$$

Thus the subsequence $s_0, s_2, s_4, \dots,$ is decreasing.

Next observe that

$$s_1=1-\frac{1}{2}>0,\ \ s_3=s_1+\frac{1}{3}-\frac{1}{4}>s_1,$$

$$s_{2n+3}=s_{2n+1}+\frac{1}{2n+3}-\frac{1}{2n+4}>s_{2n+1}.$$

Thus, the subsequence $s_1, s_3, s_5, \dots,$ is increasing. Now observe that

$$s_{2n+2}-s_{2n+1}=\frac{1}{2n+3}>0.$$

Hence

$$s_0>s_{2n+2}>s_{2n+1}\geq s_1.$$

This proves that the increasing subsequence (s_{2n+1}) is also bounded above and the decreasing sequence (s_{2n+2}) is bounded below. Hence these two subsequences are convergent and since

$$\lim_n(s_{2n+2}-s_{2n+1})=\lim_n\frac{1}{2n+3}=0$$

we deduce that they converge to the same real number. This implies that the full sequence $(s_n)_{n\geq 0}$ converges to the same number; see Exercise 4.23.

The sum of this alternating series is $\ln 2$, but the proof of this fact is more involved an requires the full strength of the calculus techniques; see Example 9.52. □

4.7. Power series

Definition 4.52. A *power series* in the variable x and real coefficients $a_0, a_1, a_2, \dots$ is a series of the form

$$s(x)=a_0+a_1x+a_2x^2+\cdots.$$

The *domain of convergence* of the power series is the set of real numbers x such that the corresponding series $s(x)$ is convergent. □

Proposition 4.53. *Consider a power series in the variable x with real coefficients*

$$s(x)=a_0+a_1x+a_2x^2+\cdots.$$

Suppose that the nonzero real number x_0 is in the domain of convergence of the series. Then for any real number x such that $|x|<|x_0|$ the series $s(x)$ is absolutely *convergent.*

Proof. Since the series

$$a_0 + a_1 x_0 + a_2 x_0^2 + \cdots$$

is convergent, the sequence $(a_n x_0^n)$ converges to zero. In particular, this sequence is bounded and thus there exists a positive constant C such that

$$|a_n x_0^n| < C, \ \ \forall n = 0, 1, 2, \dots .$$

We set

$$r := \frac{|x|}{|x_0|}$$

and we observe that $0 \leq r < 1$. Next we notice that

$$|a_n x^n| = |a_n x_0^n| \frac{|x|^n}{|x_0|^n} = |a_n x_0^n| r^n < C r^n, \ \ \forall n.$$

Since $0 \leq r < 1$ we deduce that the positive geometric series

$$C + Cr + Cr^2 + \cdots$$

is convergent. The comparison principle then implies that the series

$$|a_0| + |a_1 x| + |a_2 x^2| + \cdots$$

is also convergent. □

The above result has a very important consequence whose proof is left to you as an exercise.

Corollary 4.54. *Consider a power series in the variable x and real coefficients*

$$s(x) = a_0 + a_1 x + a_2 x^2 + \cdots .$$

We denote by D the domain of convergence of the series. We set

$$R := \begin{cases} \sup D, & \text{if } D \text{ is bounded above,} \\ \infty, & \text{if } D \text{ is not bounded above.} \end{cases} \tag{4.28}$$

Then the following hold.

(i) $R \geq 0$.

(ii) *If x is a real number such that $|x| < R$, then the series $s(x)$ is absolutely convergent.*

(iii) *If x is a real number such that $|x| > R$, then the series $s(x)$ is divergent.*

Definition 4.55. The quantity R defined in (4.28) is called the *radius of convergence* of the power series $s(x)$. □

4.8. Some fundamental sequences and series

$$\lim_{n\to\infty}\frac{C}{n}=0,\;\;\forall C>0.$$

$$\lim_{n\to\infty}Cn=\infty,\;\;\forall C>0.$$

$$\lim_{n\to\infty}r^n=\lim_{n\to\infty}\frac{1}{a^n}=0,\;\;\forall r\in(0,1),\;\;\forall a>1.$$

$$\lim_{n\to\infty}\frac{n}{r^n}=0,\;\;\forall r>1.$$

$$\lim_{n\to\infty}r^{\frac{1}{n}}=1,\;\;\forall r>0.$$

$$\lim_{n\to\infty}n^{\frac{1}{n}}=1.$$

$$\lim_{n\to\infty}\frac{r^n}{n!}=0,\;\;\forall r\in\mathbb{R}.$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

$$1+r+r^2+\cdots+r^n+\cdots=\frac{1}{1-r},\;\;\forall |r|<1.$$

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots=e.$$

$$\sum_{n\geq 1}\frac{1}{n^s}=\begin{cases}\text{convergent}, & s>1\\ \text{divergent}, & s\leq 1.\end{cases}$$

4.9. Exercises

Exercise 4.1. Prove, *using the definition*, the following equalities.

$$\lim_{n\to\infty} \frac{n}{n^2+1} = 0, \tag{a}$$

$$\lim_{n\to\infty} \frac{3n+1}{2n+5} = \frac{3}{2}, \tag{b}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0. \tag{c}$$

Exercise 4.2. Prove Proposition 4.8. □

Exercise 4.3. Prove Proposition 4.9. □

Exercise 4.4. Let $(x_n)_{n\geq 0}$ be a sequence of real numbers and $x \in \mathbb{R}$. Consider the following statements.

(i) $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that, $n > N \Rightarrow |x_n - x| < \varepsilon$.

(ii) $\exists N \in \mathbb{N}$ such that, $\forall \varepsilon > 0$, $n > N \Rightarrow |x_n - x| < \varepsilon$.

Prove that (ii) $\Rightarrow$ (i) and construct an example of sequence $(x_n)_{n\geq 1}$ and real number x satisfying (i) but not (ii). □

Exercise 4.5. (a) Prove that for any real numbers a, b we have

$$\big|\, |a| - |b| \,\big| \leq |a - b|.$$

(b) Let $(x_n)_{n\geq 0}$ be a sequence of real numbers that converges to $x \in \mathbb{R}$. Prove that

$$\lim_{n\to\infty} |x_n| = |x|.$$ □

Exercise 4.6. Compute

$$\lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \right).$$

Hint. Observe that

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} \right).$$

At this point you might want to use Exercise 3.7. □

Exercise 4.7. Compute

$$\lim_{n\to\infty} \frac{2^1 + 2^3 + 2^5 + \cdots + 2^{2n+1}}{2^{2n+3}}.$$

Hint. Use Exercise 3.7. □

Exercise 4.8. Let $X \subset \mathbb{R}$ be a bounded above set of real numbers. Denote by x^* the supremum of X. (The existence of the least upper bound of X is guaranteed by the Completeness Axiom.) Prove that there exists a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ satisfying the following properties.

(i) $x_n \in X$, $\forall n \in \mathbb{N}$.

(ii) $\lim_{n\to\infty} x_n = x^*$.

Hint. Use Proposition 2.21 and Corollary 4.11. □

Exercise 4.9. Prove the equality (4.11). □

Exercise 4.10. Let $0 < a < b$. Compute

$$\lim_{n\to\infty} \frac{a^{n+1} + b^{n+1}}{a^n + b^n}.$$ □

Exercise 4.11. (a) Let (a_n) be a sequence of positive real numbers such that $\lim_n a_n = 1$. Prove that

$$\lim_n \sqrt{a_n} = 1.$$

(b) Compute

$$\lim_{n\to\infty} \sqrt{n}\big(\sqrt{n+1} - \sqrt{n}\big).$$

Hint. Prove first that

$$\sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x+1} + \sqrt{x}}, \quad \forall x > 0.$$ □

Exercise 4.12. Prove that if $a > 0$, then

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1.$$

Hint. Consider first the case $a > 1$. Write $a^{\frac{1}{n}} = 1 + \varepsilon_n$ and then use Bernoulli's inequality. Show that the case $a < 1$ follows from the case $a > 1$. □

Exercise 4.13. Prove that for any real number x there exists an *increasing* sequence of *rational* numbers that converges to x and also a *decreasing* sequence of *rational* numbers that converges to x.

Hint. Use Proposition 3.32. □

Exercise 4.14. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers that converges to a *positive* number a. Prove that

$$\exists c > 0 \text{ such that } \forall n \in \mathbb{N} \;\; a_n > c.$$

Hint. Argue by contradiction. □

Exercise 4.15. Let $k \in \mathbb{N}$ and suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers that converges to a *positive* number a.

(a) Using Exercise 4.14 prove that there exists $r > 0$ such that $a_n > r$, $\forall n$, so that $a_n^{\frac{1}{k}} > r^{\frac{1}{k}}$, $\forall n$.

(b) Prove that there exists a constant $M > 0$ such that

$$\left| a_n^{\frac{1}{k}} - a^{\frac{1}{k}} \right| \leq M|a_n - a|, \quad \forall n \in \mathbb{N}.$$

Hint. Set $b_n := a_n^{\frac{1}{k}}$, $b := a^{\frac{1}{k}}$ and use the equality (3.10) to deduce.

$$a_n - a = b_n^k - b^k = (b_n - b)(b_n^{k-1} + b_n^{k-2}b + \cdots + b_n b^{k-2} + b^{k-1})$$

which implies

$$|b_n - b| = \frac{|a_n - a|}{b_n^{k-1} + b_n^{k-2}b + \cdots + b_n b^{k-2} + b^{k-1}}.$$

Now use part (a).

(c) Show that

$$\lim_n a_n^{\frac{1}{k}} = a^{\frac{1}{k}}.$$

(d) Show that if $r \in \mathbb{Q}$, then

$$\lim_n a_n^r = a^r.$$

□

Exercise 4.16. Ler $r > 1$ and $k \in \mathbb{N}$. Prove that

$$\lim_{n\to\infty} r^n = \infty.$$

and

$$\lim_{n\to\infty} \frac{n^k}{r^n} = 0.$$

Hint. Let $a = r^{\frac{1}{k}}$. Then

$$\frac{n^k}{r^n} = \left(\frac{n}{a^n}\right)^k.$$

□

Exercise 4.17. Compute

$$\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n.$$

□

Exercise 4.18. (a) Using Example 4.23 as inspiration prove that the sequence

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

is increasing.

(b) Prove that the Euler number e satisfies the inequalities

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}, \quad \forall n \in \mathbb{N}.$$

Deduce from the above inequalities that $2 < e < 3$. □

Exercise 4.19. Consider the sequence (x_n) defined by the recurrence

$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2 + x_n}, \quad \forall n \in \mathbb{N}.$$

Thus

$$x_2 = \sqrt{2 + \sqrt{2}}, \quad x_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \quad x_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}, \ldots.$$

(a) Prove by induction that the sequence (x_n) is increasing.

(b) Prove by induction that $x_n < \sqrt{2} + 1$, $\forall n \in \mathbb{N}$.

(c) Find $\lim_{n\to\infty} x_n$.

Hint. Consider the function $f:(0,\infty)\to(0,\infty)$, $f(x)=\sqrt{2+x}$ and prove that

$$0<x<y\Rightarrow f(x)<f(y) \ \text{ and } \ x>0\wedge x=f(x)\Longleftrightarrow x=\sqrt{2}.$$ □

Exercise 4.20. Fix $a>0$, $a\neq 1$ and define $f:(0,\infty)\to(0,\infty)$ by

$$f(x)=\frac{1}{2}\left(x+\frac{a}{x}\right)=\frac{x^2+a}{2x}.$$

Consider the sequence of positive real numbers $(x_n)_{n\geq 1}$ defined by the recurrence

$$x_1=1,\ \ x_{n+1}=f(x_n),\ \forall n\in\mathbb{N}.$$

Use the strategy employed in Example 4.25 to show that

$$\lim_{n\to\infty}x_n=\sqrt{a}.$$ □

Exercise 4.21 (Gauss)**.** Let a_0,b_0 be two real numbers such that

$$0<a_0<b_0.$$

Define inductively

$$a_1:=\sqrt{a_0b_0},\ \ b_1=\frac{a_0+b_0}{2},$$

$$a_{n+1}=\sqrt{a_nb_n},\ \ b_{n+1}=\frac{a_n+b_n}{2}.$$

(a) Prove by induction that

$$a_1\leq a_2\leq\cdots\leq a_n\leq b_n\leq\cdots\leq b_2\leq b_1.$$

(b) Prove that the sequences (a_n) and (b_n) are convergent and

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n.$$

Hint: For part (a) use Exercise 3.5. For part (b) use Weierstrass' Theorem on the convergence of bounded monotone sequences, Theorem 4.22. □

Exercise 4.22. Establish the convergence or divergence of the sequence

$$a_n=\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n},\ \ n\in\mathbb{N}.$$ □

Exercise 4.23. Let (a_n) be a sequence of real numbers. For each $n\in\mathbb{N}$ we set

$$b_n:=a_{2n-1},\ \ c_n:=a_{2n}.$$

Then the following statements are equivalent.

(i) The sequence (a_n) is convergent and its limit is $a\in\mathbb{R}$.

(ii) The subsequences $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ converge to the same limit a.

□

Exercise 4.24. Suppose $(a_n)_{n\in\mathbb{N}}$ is a *contractive* sequence of real numbers, i.e., there exists $r \in (0,1)$ such that

$$|a_n - a_{n+1}| < r|a_n - a_{n-1}|, \;\; \forall n \in \mathbb{N}, n \geq 2.$$

Prove that the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent.

Hint. Use the identity

$$\frac{1-r^m}{1-r} = 1 + r + \cdots + r^{m-1}, \;\; \forall m \in \mathbb{N}$$

and Cauchy's theorem Theorem 4.33. □

Exercise 4.25. Consider the sequence of positive real numbers $(x_n)_{n\geq 1}$ defined by the recurrence

$$x_1 = 1, \;\; x_{n+1} = 1 + \frac{1}{x_n}, \;\; \forall n \in \mathbb{N}.$$

Thus

$$x_2 = 1+1, \;\; x_3 = 1 + \frac{1}{1+1} = \frac{3}{2}, \;\; x_4 = 1 + \frac{1}{1+\frac{1}{1+1}} = 1 + \frac{2}{3} = \frac{5}{3},$$

$$x_5 = 1 + \cfrac{1}{1 + \cfrac{1}{1+\frac{1}{1+1}}}, \;\; x_6 = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1+1}}}} \cdots$$

(a) Prove that

$$x_1 < x_3 < \cdots < x_{2n+1} < x_{2n+2} < x_{2n} < \cdots < x_2, \;\; \forall n \geq 1.$$

(b) Prove that for $n \geq 3$ we have

$$|x_{n+1} - x_n| \leq \frac{|x_n - x_{n-1}|}{x_3} = \frac{2}{3}|x_n - x_{n-1}|.$$

(c) Conclude that the sequence (x_n) is convergent and find its limit. (**Hint:** Use Exercise 4.24.) □

Exercise 4.26. If $a_1 < a_2$

$$a_{n+2} = \frac{1}{2}\big(\, a_{n+1} + a_n \,\big), \;\; \forall n \in \mathbb{N}$$

show that the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent.

Hint. Use Exercise 4.24. □

Exercise 4.27. Consider a sequence of positive numbers $(x_n)_{n\geq 1}$ satisfying the recurrence relation

$$x_{n+1} = \frac{1}{2+x_n}, \;\; \forall n \in \mathbb{N}.$$

Show that $(x_n)_{n\in\mathbb{N}}$ is a contractive sequence (Exercise 4.24) and then compute its limit. □

Exercise 4.28. Find all the limit points (see Definition 4.30) of the sequence

$$a_n = (-1)^n \frac{n-1}{n}.$$ □

Exercise 4.29. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers, i.e.,

$$\exists C \in \mathbb{R}: \quad |a_n| \leq C, \quad \forall n.$$

For any $k \in \mathbb{N}$ we set

$$b_k := \sup\{a_n; \;\; n \geq k\}.$$

(a) Show that the sequence $(b_k)_{k\in\mathbb{N}}$ is *nonincreasing* and conclude that it is convergent. Denote by b its limit.

(b) Show that b is a limit point of the sequence $(a_n)_{n\in\mathbb{N}}$, i.e., there exists a subsequence $(a_{n_k})_{k\geq 1}$ of $(a_n)_{n\geq 1}$ such that

$$\lim_{k\to\infty} a_{n_k} = b.$$

(c) Show that if α is a limit point of the sequence (a_n), then $\alpha \leq b$.

The number b is called the *superior limit* of the sequence (a_n) and it is denoted by $\limsup_n a_n$. The above exercise shows that the superior limit is the largest limit point of a bounded sequence. □

Exercise 4.30. Prove Proposition 4.36. □

Exercise 4.31. Prove that if $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$ are convergent series of real numbers and $\alpha, \beta \in \mathbb{R}$, then the series $\sum_{n\geq 0}(\alpha a_n + \beta b_n)$ is convergent and

$$\sum_{n\geq 0}(\alpha a_n + \beta b_n) = \alpha \sum_{n\geq 0} a_n + \beta \sum_{n\geq 0} b_n.$$ □

Exercise 4.32. Can you give an example of convergent series $\sum_{n\geq 0} a_n$ and a divergent series $\sum_{n\geq 0} b_n$ such that $\sum_{n\geq 0}(a_n + b_n)$ is convergent? Explain. □

Exercise 4.33. Prove Corollary 4.43. □

Exercise 4.34. Consider the sequence

$$a_n = \frac{n^3 + 2n^2 + 2n + 4}{n^5 + n^4 + 7n^2 + 1}, \quad n \geq 0.$$

Prove that the series

$$\sum_{n\geq 0} a_n$$

is absolutely convergent.

Hint. Example 4.40(b) and Corollary 4.43. □

Exercise 4.35 (Leibniz)**.** Suppose that (a_n) is a decreasing sequence of positive real numbers such that

$$\lim_{n\to\infty} a_n = 0.$$

Prove that the series

$$\sum_{n\geq 0} (-1)^n a_n$$

is convergent.

Hint. Imitate the strategy in Example 4.51. □

Exercise 4.36 (Cauchy)**.** Suppose that $(a_n)_{n\geq 0}$ is a decreasing sequence of positive numbers that converges to 0. Prove that the series

$$\sum_{n\geq 0} a_n$$

converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots$$

converges.

Hint. Imitate the strategy employed in Example 4.40. □

Exercise 4.37. We consider the power series series

$$\sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Suppose that there exists $C > 0$ such that $|a_n| \leq C$, $\forall n$. Show that the radius of convergence of the series

$$\sum_{n\geq 0} a_n x^n$$

is ≥ 1. □

Exercise 4.38. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of integers such that $0 \leq a_n \leq 9$ for any $n \in \mathbb{N}$, i.e.,

$$a_n \in \{0, 1, 2, \ldots, 9\}, \;\; \forall n \in \mathbb{N}.$$

Show that the series

$$\sum_{n\geq 1} a_n 10^{-n} = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots$$

is convergent.

(b) Compute the sum of the above series in the two special special cases

$$a_n = 7, \;\; \forall n \in \mathbb{N},$$

and

$$a_n = \begin{cases} 1, & n \text{ is odd} \\ 2, & n, \text{ is even.} \end{cases}$$

In each case, express the sum in decimal form.

(c) Prove that for any $x \in [0, 1]$ there exists a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ such that

$$a_n \in \{0, 1, 2, \ldots, 9\}, \;\; \forall n \in \mathbb{N},$$

and

$$x = \sum_{n\geq 1} a_n 10^{-n}.$$ □

Exercise 4.39. Prove Corollary 4.54. □

4.10. Exercises for extra-credit

Exercise* 4.1. Fix rational numbers a, b such that $1 < a < b$.

(a) Prove that

$$\lim_{n\to\infty} \frac{(2n)^b}{(2n+1)^a} = \infty.$$

(b) Prove that the series

$$\frac{1}{1^a} + \frac{1}{2^b} + \frac{1}{3^a} + \frac{1}{4^b} + \cdots$$

is convergent. $\square$

Exercise* 4.2. Consider two series of real numbers $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$. For each non-negative integer n define

$$c_n := a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$$

Prove that the if the series $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$ are *absolutely* convergent, then the series

$$\sum_{n\geq 0} c_n$$

is *absolutely* convergent and its sum is the product of the sums of the series $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$, i.e.,

$$\lim_{n\to\infty} \sum_{k=0}^{n} c_n = \left(\lim_{n\to\infty} \sum_{j=0}^{n} a_j \right) \cdot \left(\lim_{n\to\infty} \sum_{k=0}^{n} b_k \right).$$

The series $\sum_{n\geq 0} c_n$ constructed above is called the *Cauchy product* of the series $\sum_{n\geq 0} a_n$ and $\sum_{n\geq 0} b_n$.

Hint: Consider first the special case $a_n, b_n \geq 0$, $\forall n$. Set

$$A_n := \sum_{j=0}^{n} a_j, \quad B_n := \sum_{k=0}^{n} b_k, \quad C_n = \sum_{\ell=0}^{n} c_\ell.$$

Prove that

$$\lim_{n\to\infty} (C_n - A_n B_n) = 0.$$

$\square$

Exercise* 4.3. Let (a_n) be a convergent sequence of real numbers. Form the new sequence (c_n) defined by the rule

$$c_n := \frac{a_1 + \cdots + a_n}{n}$$

Show that (c_n) is convergent and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n.$$

$\square$

Exercise* 4.4. Let the two given sequences

$$a_0, a_1, a_2, \ldots,$$

$$b_0, b_1, b_2, \ldots$$

satisfy the conditions

$$b_n > 0, \;\; \forall n \geq 0, \tag{4.29a}$$

$$b_0 + b_1 + b_2 + \cdots + b_n + \cdots = \infty, \tag{4.29b}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = s. \tag{4.29c}$$

Prove that

$$\lim_{n\to\infty} \frac{a_0 + a_1 + \cdots + a_n}{b_0 + b_1 + \cdots + b_n} = s.$$ □

Exercise* 4.5. Suppose that $(p_n)_{n\geq 1}$ is a sequence of *positive* real numbers, and $(x_n)_{n\geq 1}$ is a sequence of real numbers. For $n \in \mathbb{N}$ we set

$$b_n := p_1 + \cdots + p_n, \;\; s_n := x_1 + \cdots + x_n.$$

Suppose that

$$\lim_{n\to\infty} b_n = \infty.$$

Prove that if the series

$$\sum_{n\geq 1} \frac{x_n}{b_n}$$

is convergent, then

$$\lim_{n\to\infty} \frac{s_n}{b_n} = 0.$$ □

Exercise* 4.6 (Fekete)**.** Suppose that the sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ satisfies the subadditivity condition

$$a_{m+n} \leq a_m + a_n, \;\; \forall m, n \in \mathbb{N}.$$

Prove that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}.$$ □

Exercise* 4.7. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n+1}x_{n-1} = 1, \;\; \forall n \in \mathbb{N}.$$

Prove that there exists $a \in \mathbb{R}$ such that

$$x_{n+1} = ax_n - x_{n-1}, \;\; \forall n \in \mathbb{N}.$$ □

Exercise* 4.8. Suppose that a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ satisfies

$$0 < a_n < a_{2n} + a_{2n+1}, \;\; \forall n \in \mathbb{N}.$$

Prove that the series $\sum_{n\geq 1} a_n$ is divergent. □

Exercise* 4.9. Suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers such that the series

$$\sum_{n\in\mathbb{N}} x_n$$

is convergent and its sum is S. Prove that for any bijection $\varphi:\mathbb{N}\to\mathbb{N}$ the series

$$\sum_{n\in\mathbb{N}} x_{\varphi(n)}$$

is also convergent and its sum is also S. □

Exercise* 4.10. Suppose that the series of real numbers

$$\sum_{n\in\mathbb{N}} x_n$$

is convergent, but *not absolutely convergent*. Prove that for any real number S there exists a bijection $\varphi:\mathbb{N}\to\mathbb{N}$ such that the series

$$\sum_{n\in\mathbb{N}} x_{\varphi(n)}$$

is convergent and its sum is S. □

Exercise* 4.11. Suppose that $(a_n)_{n\geq 1}$ is a decreasing sequence of positive real numbers that converges to 0 and satisfies the inequalities

$$a_n \leq a_{n+1} + a_{n^2},\ \ \forall n\geq 1.$$

Prove that the series

$$\sum_{n\geq 1} a_n$$

is divergent. □

Exercise* 4.12. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a a function satisfying the following conditions

$$f(x+y) = f(x)+f(y),\ \ \forall x,y\in\mathbb{R}. \tag{4.30a}$$

$$f(xy) = f(x)f(y),\ \ \forall x,y\in\mathbb{R}. \tag{4.30b}$$

$$f(1)\neq 0. \tag{4.30c}$$

Prove that the following hold.

(i) $f(0)=0$, $f(1)=1$.

(ii) $f(n)=n$, $\forall n\in\mathbb{N}$.

(iii) $f(m)=m$, $\forall m\in\mathbb{Z}$.

(iv) $f(q)=q$, $\forall q\in\mathbb{Q}$.

(v) If $x,y\in\mathbb{R}$ and $x<y$, then $f(x)<f(y)$.

(vi) $f(x)=x$, $\forall x\in\mathbb{R}$.

Chapter 5

Limits of functions

5.1. Definition and basic properties

Let X be a nonempty subset of $\mathbb{R}$. A real number c is called a *cluster point of* X if there exists a sequence (x_n) of real numbers with the following properties.

(i) $x_n \in X$, $\forall n \in \mathbb{N}$.

(ii) $x_n \neq c$, $\forall n \in \mathbb{N}$.

(iii) $\lim_n x_n = c$.

Example 5.1. (a) If $A = (0,1)$, then 0 and 1 are cluster points of A, although they are not in A. Indeed, the sequence $a_n = \frac{1}{n+1}$, $n \in \mathbb{N}$ consists of elements of $(0,1)$ and $a_n \to 0$. Similarly, the sequence $b_n = 1 - \frac{1}{n+1}$ consists of points in $(0,1)$ and $b_n \to 1$. Observe that every point in $(0,1)$ is also a cluster point of $(0,1)$.

(b) Any real number is a cluster point of the set $\mathbb{Q}$ of rational numbers. □

Definition 5.2. Let $X \subset \mathbb{R}$. Suppose that c is a cluster point of X and $f : X \to \mathbb{R}$ is a real valued function defined on X. We say that the limit of f at c is the real number A, and we write this

$$\lim_{x \to c} f(x) = A,$$

if the following holds:

$$\forall \varepsilon > 0 \;\; \exists \delta = \delta(\varepsilon) > 0 : \;\; \forall x \in X : \;\; 0 < |x - c| < \delta \Rightarrow |f(x) - A| < \varepsilon. \tag{5.1}$$

□

An alternate viewpoint. Recall that a *neighborhood* of a point a is an open interval that contains a inside. For example, the open interval $(0,3)$ is a neighborhood of 1. We denote by $\mathcal{N}_a$ the collection of all neighborhoods of a. Thus, a statement of the form $U \in \mathcal{N}_a$ signifies that U is an open interval that contains a. A *symmetric neighborhood* of a is a neighborhood of the form $(a - \delta, a + \delta)$, where δ is some positive number. Observe that

$$x \in (a - \delta, a + \delta) \Longleftrightarrow \operatorname{dist}(a, x) < \delta \Longleftrightarrow |x - a| < \delta.$$

Thus, to describe a symmetric neighborhood of a, it suffices to indicate a positive real number δ, and then the symmetric neighborhood is described by the condition $\operatorname{dist}(x,a) < \delta$. We denote by $\mathcal{SN}_a$ the collection of symmetric neighborhoods of a. Clearly, any symmetric neighborhood of a is also a neighborhood of a so that

$$\mathcal{SN}_a \subset \mathcal{N}_a.$$

A *deleted neighborhood* of a is a set obtained from a neighborhood of a by removing the point a. For example

$$(0,2)\setminus\{1\} = (0,1)\cup(1,2)$$

is a deleted neighborhood of 1. We denote by $\mathcal{N}_a^*$ the collection of all deleted neighborhoods of a. A *symmetric deleted neighborhood* of a is a deleted neighborhood of the form

$$(a-r,a+r)\setminus\{a\} = (a-r,a)\cup(a,a+r).$$

We denote by $\mathcal{SN}_a^*$ the collection of deleted symmetric neighborhoods of a. Clearly

$$\mathcal{SN}_a^* \subset \mathcal{S}_a^*.$$

Observe that the definition (5.1) is equivalent with the following statement

$$\forall U\in\mathcal{SN}_a \;\; \exists V\in\mathcal{SN}_c^* \;\; \forall x\in X: \;\; x\in V \Rightarrow f(x)\in U. \tag{5.2}$$

Indeed, we can rephrase (5.1) in the following equivalent way: for any symmetric neighborhood U of A of the form $(A-\varepsilon, A+\varepsilon)$, there exists a deleted symmetric neighborhood V of c of the form $(c-\delta,c+\delta)\setminus\{c\}$ such that for any $x\in V$ we have $f(x)\in U$. That is precisely the content of (5.2).

The proof of the next result is left to you as an exercise.

Proposition 5.3. *Let $f: X\to\mathbb{R}$ be a function defined on a set $X\subset\mathbb{R}$ and c a cluster point of X. Then the following statements are equivalent.*

(i) $\lim_{x\to c} f(x) = A$, *i.e., f satisfies (5.1) or (5.2).*

(ii)
$$\forall U\in\mathcal{N}_A, \;\; \exists V\in\mathcal{N}_c^* \;\; \text{such that} \;\; \forall x\in X: x\in V\Rightarrow f(x)\in U. \tag{5.3}$$

□

The following very useful result reduces the study of limits of functions to the study of a concept we are already familiar with, namely the concept of limits of sequences.

Theorem 5.4. *Let c be a cluster point of the set $X\subset\mathbb{R}$ and $f: X\to\mathbb{R}$ a real valued function on X. The following statements are equivalent.*

(i) $\lim_{x\to c} f(x) = A\in\mathbb{R}$.

(ii) *For any sequence $(x_n)_{n\in\mathbb{N}}$ in $X\setminus\{c\}$ such that $x_n\to c$, we have $\lim_n f(x_n) = A$.*

Proof. (i) $\Rightarrow$ (ii). We know that $\lim_{x\to c} f(x) = A$ and we have to show that if (x_n) is a sequence in $X\setminus\{c\}$ that converges to c, then the sequence $(\, f(x_n)\,)$ converges to A. In other words, given the above sequence (x_n) we have to show that

$$\forall\varepsilon>0 \;\; \exists N=N(\varepsilon) \;\; \forall n\in\mathbb{N}: \;\; n>N(\varepsilon)\Rightarrow |f(x_n)-A|<\varepsilon.$$

Let $\varepsilon>0$. We deduce from (5.1) that there exists $\delta(\varepsilon)>0$ such that

$$\forall x\in X: \;\; 0<|x-c|<\delta\Rightarrow|f(x)-A|<\varepsilon. \tag{5.4}$$

Since $x_n\to c$, there exists $N=N(\delta(\varepsilon))$ such that

$$0<|x_n-c|<\delta, \;\; \forall n>N.$$

Using (5.4) we deduce that for any $n>N(\delta(\varepsilon))$ we have $|f(x_n)-A|<\varepsilon$. This proves the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i) We know that for any sequence (x_n) in $X \setminus \{c\}$ that converges to c, the sequence $(\, f(x_n)\,)$ converges to A and we have to prove (5.1), i.e.,

$$\forall \varepsilon > 0 \;\; \exists \delta = \delta(\varepsilon) > 0 : \;\; \forall x \in X : \;\; 0 < |x - c| < \delta \Rightarrow |f(x) - A| < \varepsilon. \tag{5.5}$$

We argue by contradiction and we assume that (5.5) is false, so that its opposite is true, i.e.,

$$\exists \varepsilon_0 > 0 : \;\; \forall \delta > 0, \;\; \exists x = x(\delta) \in X, \;\; 0 < |x(\delta) - c| < \delta \;\text{ and }\; |f(\, x(\delta)\,) - A| \geq \varepsilon_0. \tag{5.6}$$

From (5.6) we deduce that for any δ of the form $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, there exists $x_n = x(1/n) \in X$ such that

$$0 < |x_n - c| < \frac{1}{n} \;\wedge\; |f(\, x_n\,) - A| \geq \varepsilon_0.$$

We have thus produced a sequence (x_n) in X such that

$$0 < \operatorname{dist}(x_n, c) < \frac{1}{n} \;\wedge\; \operatorname{dist}(f(x_n), A) \geq \varepsilon_0.$$

Thus, (x_n) is a sequence in $X \setminus \{c\}$ that converges to c, but the sequence $(\, f(x_n)\,)$ does not converge to A. $\square$

Using Proposition 4.15 we obtain the following immediate consequence.

Corollary 5.5. *Let $f, g : X \to \mathbb{R}$ be two functions defined on the same subset $X \subset \mathbb{R}$ and c a cluster point of X. Suppose additionally that*

$$\lim_{x \to c} f(x) = A \;\; \textit{and} \;\; \lim_{x \to c} g(x) = B.$$

Then the following hold.

(i)
$$\lim_{x \to c} \big(\, f(x) + g(x)\,\big) = A + B, \;\; \lim_{x \to c} \lambda f(x) = \lambda A, \;\; \forall \lambda \in \mathbb{R}.$$

(ii)
$$\lim_{x \to c} f(x) g(x) = AB.$$

(iii) *If $B \neq 0$ and $g(x) \neq 0$, $\forall x \in X$, then*
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

$\square$

Example 5.6. (a) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$. Then for any $c \in \mathbb{R}$ we have

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$

(b) Let $m \in \mathbb{N}$ and define $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^m$. Corollary 5.5 implies that

$$\lim_{x \to c} f(x) = \lim_{x \to c} x^m = c^m.$$

Thus,

$$\lim_{x \to 3} x^2 = 3^2 = 9.$$

(c) Let $m \in \mathbb{N}$ and define $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{-m} = \frac{1}{x^m}$. Corollary 5.5 implies that for any $c > 0$ we have

$$\lim_{x \to c} x^{-m} = \lim_{x \to c} \frac{1}{x^m} = c^{-m}.$$

(d) Let $m, k \in \mathbb{N}$ and define $f:(0,\infty)\to\mathbb{R}$, $f(x)=x^{\frac{m}{k}}$. We want to show that for any $c>0$ we have

$$\lim_{x\to c} x^{\frac{m}{k}} = c^{\frac{m}{k}}. \tag{5.7}$$

We rely on Theorem 5.4. Suppose that (x_n) is a sequence of positive numbers such that $x_n\to c$ and $x_n\neq c$, $\forall n$. We have to show that

$$\lim_n x_n^{\frac{m}{k}} = c^{\frac{m}{k}}.$$

Using Exercise 4.15, we deduce that

$$\lim_n x_n^{\frac{1}{k}} = c^{\frac{1}{k}}.$$

Thus,

$$\lim_n x_n^{\frac{m}{k}} = \lim_n \big(x_n^{\frac{1}{k}} \big)^m = \big(c^{\frac{1}{k}} \big)^m = c^{\frac{m}{k}}.$$

Thus,

$$\lim_{x\to c} x^r = c^r,\ \ \forall r\in\mathbb{Q},\ \ r>0.$$

The above equality obviously holds if $r=0$. If $r<0$, then $x^{-r}=\frac{1}{x^r}$ and we deduce

$$\lim_{x\to c} x^r = c^r,\ \ \forall c>0,\ r\in\mathbb{Q}. \tag{5.8}$$

□

Proposition 5.7. *Let $f, g : X\to\mathbb{R}$ be two functions defined on the same subset $X\subset\mathbb{R}$. Suppose that c is a cluster point of X and*

$$\lim_{x\to c} f(x) = A,\ \ \lim_{x\to c} g(x) = B \ \ \textit{and}\ \ A<B.$$

Then there exists a neighborhood U of c such that $f(x)<g(x)$, $\forall x\in U\cap X$, $x\neq c$.

Proof. Fix a positive number ε such that $3\varepsilon < B-A$. In other words, ε is smaller than one third of the distance from A to B. In particular, $A+\varepsilon<B-\varepsilon$ because

$$B-\varepsilon-(A+\varepsilon) = B-A-2\varepsilon > 3\varepsilon-2\varepsilon>0.$$

Since $\lim_{x\to c} f(x)=A$, there exists $\delta=\delta_f(\varepsilon)>0$ such that

$$\forall x\in X:\ \ 0<|x-c|<\delta_f \Rightarrow A-\varepsilon<f(x)<A+\varepsilon.$$

Since $\lim_{x\to c} g(x)=B$, there exists $\delta=\delta_g(\varepsilon)>0$ such that

$$\forall x\in X:\ \ 0<|x-c|<\delta_g \Rightarrow B-\varepsilon<f(x)<B+\varepsilon.$$

Let $\delta_0<\min\{\delta_f,\delta_g\}$ and define

$$U := (c-\delta_0, c+\delta_0).$$

If $x\in U\cap X$, $x\neq c$, then

$$0<|x-c|<\delta_0<\min\{\delta_f,\delta_g\} \Rightarrow f(x)<A+\varepsilon<B-\varepsilon<g(x).$$

□

5.2. Exponentials and logarithms

In this section we want to give a meaning to the exponential a^x where a is a positive real number and x is an arbitrary real number. The case $a = 1$ is trivial: we define $1^x = 1$, $\forall x \in \mathbb{R}$.

We consider next the case $a > 1$. In Exercise 3.14 we defined a^r for any $r \in \mathbb{Q}$ and we showed that

$$a^{r_1+r_2} = a^{r_1} \cdot a^{r_2}, \quad a^{r_1-r_2} = \frac{a^{r_1}}{a^{r_2}}, \quad \left(a^{r_1}\right)^{r_2} = a^{r_1 r_2}, \quad \forall r_1, r_2 \in \mathbb{Q}. \tag{5.9}$$

We will use these facts to define a^x for any $x \in \mathbb{R}$. This will require several auxiliary results.

Lemma 5.8. *If $a > 1$, then for any rational numbers r_1, r_2 we have*

$$r_1 < r_2 \Rightarrow a^{r_1} < a^{r_2}.$$

Proof. We will use the fact that if $x, y > 0$ and $n \in \mathbb{N}$, then

$$x < y \Longleftrightarrow x^n < y^n.$$

Since $a > 1$ we deduce that $a^{\frac{1}{n}} > 1$ because

$$\left(a^{\frac{1}{n}}\right)^n = a > 1 = 1^n.$$

Thus,

$$a^{\frac{m}{n}} > 1, \quad \forall m, n \in \mathbb{N}$$

that is,

$$a^r > 1, \quad \forall r \in \mathbb{Q}, \quad r > 0.$$

Suppose that $r_1 < r_2$. Then the above inequality implies that

$$\frac{a^{r_2}}{a^{r_1}} \stackrel{(5.9)}{=} a^{r_2-r_1} > 1$$

because $r = r_2 - r_1$ is a positive rational number. □

Lemma 5.9. *Let $a > 1$ and $r_0 \in \mathbb{Q}$. Then*

$$\lim_{\mathbb{Q} \ni r \to r_0} a^r = a^{r_0}.$$

Proof. We first consider the case $r_0 = 0$, i.e., we first prove that

$$\lim_{\mathbb{Q} \ni r \to 0} a^r = 1. \tag{5.10}$$

We have to prove that, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$0 < |r| < \delta \text{ and } r \in \mathbb{Q} \Rightarrow |a^r - 1| < \varepsilon.$$

Observe first that Exercise 4.12 implies that

$$\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} a^{-\frac{1}{n}} = 1.$$

In particular, this implies that there exists $n_0 = n_0(\varepsilon) > 0$ such that, for all $n \geq n_0$, we have

$$1 - \varepsilon < a^{-\frac{1}{n}} < a^{\frac{1}{n}} < 1 + \varepsilon.$$

We set $\delta(\varepsilon) = \frac{1}{n_0(\varepsilon)}$. If $0 < |r| < \delta(\varepsilon)$ and $r \in \mathbb{Q}$, then $-\frac{1}{n_0(\varepsilon)} < r < \frac{1}{n_0(\varepsilon)}$ and we deduce from Lemma 5.8 that

$$1 - \varepsilon < a^{-\frac{1}{n_0(\varepsilon)}} < a^r < a^{\frac{1}{n_0(\varepsilon)}} < 1 + \varepsilon \Rightarrow 1 - \varepsilon < a^r < 1 + \varepsilon \Rightarrow |a^r - 1| < \varepsilon.$$

This proves (5.10). To deal with the general case, let $r_0 \in \mathbb{Q}$. If r_n is a sequence of rational numbers $r_n \to r_0$, then

$$a^{r_n} = a^{r_0} a^{r_n - r_0}.$$

Since $r_n - r_0 \to 0$, we deduce from (5.10) that $a^{r_n - r_0} \to 1$ and thus, $a^{r_n} = a^{r_0} a^{r_n - r_0} \to a^{r_0}$. The conclusion now follows from Theorem 5.4. $\square$

Proposition 5.10. *Let $a > 1$ and $x \in \mathbb{R}$. We set*

$$\mathbb{Q}_{<x} := \big\{ r \in \mathbb{Q}, \;\; r < x \big\}, \;\; \mathbb{Q}_{>x} := \big\{ r \in \mathbb{Q}, \;\; r > x \big\}$$

$$s_x = \sup_{r \in \mathbb{Q}_{<x}} a^r, \;\; i_x = \inf_{r \in \mathbb{Q}_{>x}} a^r.$$

Then $s_x = i_x$. Moreover, if x is rational, then $s_x = i_x = a^x$.

Proof. Observe first that the set $\{a^r; \;\; r \in \mathbb{Q}_{<x}\}$ is bounded above. Indeed, if we choose a rational number $R > x$, then Lemma 5.8 implies that $a^r < a^R$ for any rational number $r < x$. A similar argument shows that the set $\{a^r; \;\; r \in \mathbb{Q}_{>x}\}$ is bounded below and we have

$$s_x \leq i_x.$$

Observe that for any rational numbers r_1, r_2 such that $r_1 < x < r_2$, we have

$$a^{r_1} \leq s_x \leq i_x \leq a^{r_2}.$$

Hence,

$$1 \leq \frac{i_x}{s_x} \leq \frac{a^{r_2}}{s_x} \leq \frac{a^{r_2}}{a^{r_1}} = a^{r_2 - r_1}.$$

Now choose two sequences $(r'_n) \subset \mathbb{Q}_{<x}$ and $(r''_n) \subset \mathbb{Q}_{>x}$ such that $r'_n \to x$ and $r''_n \to x$.[1] Then

$$1 \leq \frac{s_x}{i_x} \leq a^{r''_n - r'_n}.$$

If we let $n \to \infty$ and observe that $r''_n - r'_n \to 0$, we deduce from Lemma 5.9 that

$$1 \leq \frac{s_x}{i_x} \leq \lim_{n \to \infty} a^{r''_n - r'_n} = 1 \Rightarrow s_x = i_x.$$

If $x \in \mathbb{Q}$, then the sequences r'_n and r''_nabove converge to x. Invoking Lemma 5.9 we deduce

$$s_x = \lim_n a^{r'_n} = a^x = \lim_n a^{r''_n} = i_x.$$

$\square$

Definition 5.11. For any $a > 1$ and $x \in \mathbb{R}$ we set

$$\boxed{a^x := \sup\big\{a^r; \;\; r \in \mathbb{Q}, \;\; r < x\big\} = \inf\big\{a^r; \;\; r \in \mathbb{Q}, \;\; r > x\big\}}.$$

If $b \in (0,1)$, then $\frac{1}{b} > 1$ and we set

$$b^x := \left(\frac{1}{b}\right)^{-x}.$$

$\square$

[1]The existence of such sequences was left to you as Exercise 4.13.

Lemma 5.12. *Let $a > 1$. If $x < y$, then $a^x < a^y$.*

Proof. We can find rational numbers r_1, r_2 such that

$$x < r_1 < r_2 < y.$$

Then $r_1 \in \mathbb{Q}_{>x}$ and $r_2 \in \mathbb{Q}_{<y}$ so that

$$a^x \leq a^{r_1} < a^{r_2} \leq a^y.$$

$\square$

Lemma 5.13. *Let $a > 1$ and $x \in \mathbb{R}$. If the sequence $(r_n) \subset \mathbb{Q}_{<x}$ converges to x, then*

$$\lim_{n\to\infty} a^{r_n} \to a^x.$$

Proof. We have

$$a^x = \sup_{r\in\mathbb{Q}_{<x}} a^r.$$

Thus, for any $\varepsilon > 0$, there exists $r_\varepsilon \in \mathbb{Q}_{<x}$ such that

$$a^x - \varepsilon < a^{r_\varepsilon} \leq a^x.$$

Since $r_n \to x$ and $r_n \in \mathbb{Q}_{<x}$, we deduce that there exists $N = N(\varepsilon)$ such that, $\forall n > N(\varepsilon)$ we have $r_\varepsilon < r_n < x$. We deduce that for all $n > N(\varepsilon)$, we have

$$a^x - \varepsilon < a^{r_\varepsilon} < a^{r_n} < a^x.$$

$\square$

Lemma 5.14. *Let $a > 0$ and $x, y > 0$. Then*

$$a^x \cdot a^y = a^{x+y}.$$

Proof. Choose sequences $(r'_n) \subset \mathbb{Q}_{<x}$ and $(r''_n) \subset \mathbb{Q}_{<y}$ such that $r'_n \to x$ and $r''_n \to y$. Lemma 5.13 implies that

$$a^{r'_n} \to a^x \quad \wedge \quad a^{r''_n} \to a^y.$$

Hence,

$$\lim_n a^{r'_n + r''_n} = \lim_n \big(a^{r'_n} \cdot a^{r''_n} \big) = \big(\lim_n a^{r'_n} \big) \cdot \big(\lim_n a^{r''_n} \big) = a^x \cdot a^y.$$

Now observe that $r'_n + r''_n \in \mathbb{Q}_{<x+y}$ and $r'_n + r''_n \to x + y$. Lemma 5.13 implies

$$\lim_n a^{r'_n + r''_n} = a^{x+y}.$$

$\square$

The proofs of our next two results are left to you as an exercise.

Lemma 5.15. *Let $a > 0$ and $x \in \mathbb{R}$. Then for any sequence of real numbers (x_n) such that $x_n \to x$ we have*

$$\lim_{n\to\infty} a^{x_n} = a^x.$$

$\square$

Lemma 5.16. *Suppose that $a, b > 0$. Then for any $x \in \mathbb{R}$ we have*

$$a^x \cdot b^x = (ab)^x. \tag{5.11}$$

$\square$

Definition 5.17. Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ be a real valued function defined on X.

(i) The function f is called *increasing* if

$$\forall x_1, x_2 \in X \ \ (x_1 < x_2) \Rightarrow \big(f(x_1) < f(x_2) \big).$$

(ii) The function f is called *decreasing* if

$$\forall x_1, x_2 \in X \;\; (x_1 < x_2) \Rightarrow \big(f(x_1) > f(x_2) \big).$$

(iii) The function f is called *nondecreasing* if

$$\forall x_1, x_2 \in X \;\; (x_1 < x_2) \Rightarrow \big(f(x_1) \leq f(x_2) \big).$$

(iv) The function f is called *nonincreasing* if

$$\forall x_1, x_2 \in X \;\; (x_1 < x_2) \Rightarrow \big(f(x_1) \geq f(x_2) \big).$$

(v) The function is called *strictly monotone* if it is either increasing or decreasing. It is called *monotone* if it is either nondecreasing or nonincreasing.

□

Theorem 5.18. *Let $a > 0$, $a \neq 1$. Consider the function $f_a : \mathbb{R} \to (0, \infty)$ given by $f(x) = a^x$. Then the following hold.*

(i) $a^{x+y} = a^x \cdot a^y$, $\forall x, y \in \mathbb{R}$.

(ii) $(a^x)^y = a^{xy}$, $\forall x, y \in \mathbb{R}$.

(iii) *The function f_a is increasing if $a > 1$, and decreasing if $a < 1$.*

(iv) *The function f is bijective.*

(v) *For any sequence of real numbers (x_n) such that $x_n \to x$ we have*

$$\lim_{n\to\infty} a^{x_n} = a^x.$$

Proof. Part (v) above is Lemma 5.15. We thus have to prove (i)-(iv). We consider first the case $a > 1$. The equality (i) is Lemma 5.14. The statement (iii) follows from Lemma 5.12.

We first prove (ii) in the special case $y \in \mathbb{Q}$. Choose a sequence $r_n \in \mathbb{Q}$ such that $r_n \to x$, $r_n \neq x$. Then (5.9) implies

$$(a^{r_n})^y = a^{r_n y}.$$

Clearly $r_n y \to xy$ and Lemma 5.15 implies that

$$\lim_n a^{r_n y} = a^{xy}.$$

On the other hand, y is rational and $a^{r_n} \to a^x$ and using (5.8) we deduce that

$$\lim_n \big(a^{r_n} \big)^y = \big(a^x \big)^y.$$

Thus,

$$\big(a^x \big)^y = \lim_n \big(a^{r_n} \big)^y = \lim_n a^{r_n y} = a^{xy}, \;\; \forall x \in \mathbb{R}, \;\; y \in \mathbb{Q}. \tag{5.12}$$

Now fix $x, y \in \mathbb{R}$ and choose a sequence of *rational* numbers $y_n \to y$, $y_n \neq y$. Then

$$\big(a^x \big)^{y_n} \stackrel{(5.12)}{=} a^{x y_n}, \;\; \forall n.$$

Using Lemma 5.15, we deduce

$$\big(a^x \big)^y = \lim_n \big(a^x \big)^{y_n} = \lim_n a^{x y_n} = a^{xy}, \;\; \forall x, y \in \mathbb{R}.$$

This proves (ii).

To prove (iv) observe that f_a is injective because it is increasing. (We recall that we are working under the assumption $a > 1$.) To prove surjectivity, fix $y \in (0, \infty)$. We have to show that there exists $x \in \mathbb{R}$ such that $a^x = y$. Define

$$S := \big\{ s \in \mathbb{R}; \;\; a^s \leq y \big\}.$$

Observe first that $S \neq \emptyset$. Indeed

$$\lim_n a^{-n} = \lim_n \frac{1}{a^n} = 0$$

so that there exists $n_0 \in \mathbb{N}$ such that $a^{-n_0} < y$, i.e., $-n_0 \in S$. Observe that S is also bounded above. Indeed

$$\lim_n a^n = \infty.$$

Hence there exists $n_1 \in \mathbb{N}$ such that $a^{n_1} > y$. If $x \geq n_1$, then $a^x \geq a^{n_1} > y$ so that $S \cap [n_1, \infty) = \emptyset$ and thus $S \subset (-\infty, n_1)$ and therefore n_1 is an upper bound for S. Set

$$x := \sup S.$$

Note that if $x' > x$, then $a^{x'} \geq y$. Indeed, if $a^{x'} < y$ then for any $s < x'$ we have $a^s < a^{x'} < y$ and thus $(-\infty, x'] \subset S$. This contradicts the fact that x is an upper bound for S.

Consider now two sequences $s'_n \to x$ and $s''_n \to x$ where $s'_n < x$ and $s''_n > x$ then

$$a^{s'_n} \leq y \leq a^{s''_n}, \ \ \forall n.$$

Letting $n \to \infty$ in the above inequalities we obtain, from Lemma 5.15, that

$$a^x \leq y \leq a^x \Longleftrightarrow a^x = y.$$

The case $a < 1$ follows from the case $a > 1$ by observing that

$$a^x = \left(\frac{1}{a}\right)^{-x}.$$

$\square$

Definition 5.19. Let $a \in (0, \infty)$, $a \neq 1$. The bijective function

$$\mathbb{R} \ni x \mapsto a^x \in (0, \infty)$$

is called the *exponential function with base* a. Its inverse is called the *logarithm to base* a and it is a function

$$\log_a : (0, \infty) \to \mathbb{R}.$$

When $a = e =$ the Euler number, we will refer to $\log_e$ as the *natural logarithm* and we will use the simpler notation log or ln. Also, we will use the notation lg for $\log_{10}$. $\square$

We have depicted below the graphs of the functions a^x and $\log_a x$ for $a = 2$ and $a = \frac{1}{2}$.

The meaning of the logarithm function answers the following question: given $a, y > 0$, $a \neq 1$, to what power do we need to raise a in order to obtain y? The answer: we need to raise a to the power $\log_a y$ in order to get y. Equivalently, $\log_a$ is uniquely determined by the following two fundamental identities

$$\log_a a^x = x \ \text{ and } \ a^{\log_a y} = y, \ \ \forall x \in \mathbb{R}, \ \ y > 0.$$

For example, $\log_2 8 = 3$ because $2^3 = 8$. Similarly $\lg 10{,}000 = 4$ since $10^4 = 10{,}000$.

Theorem 5.20. *Let $a > 0$, $a \neq 1$. Then the following hold.*

(i)

$$\log_a(y_1 y_2) = \log_a y_1 + \log_a y_2, \ \ \log_a \frac{y_1}{y_2} = \log_a y_1 - \log_a y_2, \ \ \forall y_1, y_2 > 0.$$

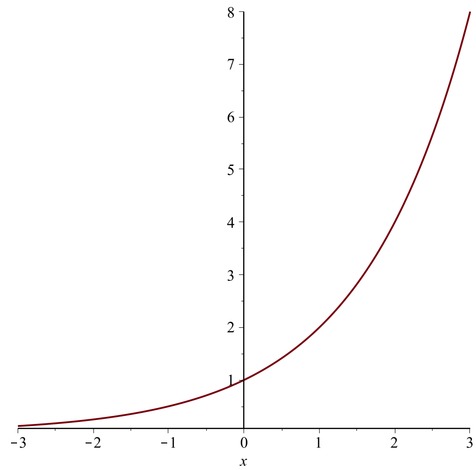

Figure 5.1. *The graph of* 2^x.

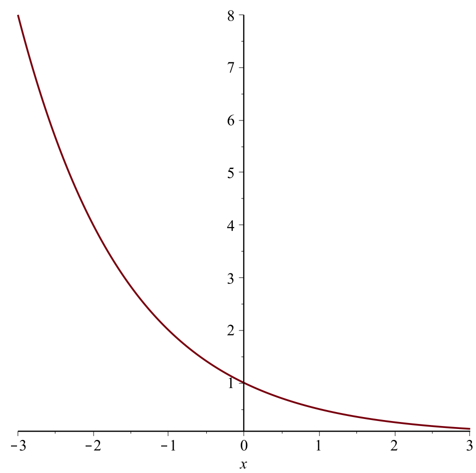

Figure 5.2. *The graph of* $\left(\frac{1}{2}\right)^x$.

(ii) $\log_a y^\alpha = \alpha \log_a y$, $\forall y > 0$, $\alpha \in \mathbb{R}$.

(iii) *If* $b > 0$ *and* $b \neq 1$, *then*

$$\log_b y = \frac{\log_a y}{\log_a b}, \quad \forall y > 0.$$

(iv) *If* $a > 1$, *then the function* $y \mapsto \log_a y$ *is increasing, while if* $a \in (0,1)$, *then the function* $y \mapsto \log_a y$ *is decreasing.*

(v) *If* $y > 0$, *then for any sequence of positive numbers* (y_n) *that converges to* y *we have*

$$\lim_{n\to\infty} \log_a y_n = \log_a y.$$

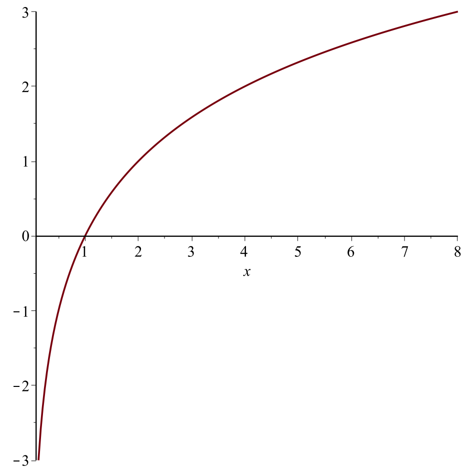

Figure 5.3. *The graph of* $\log_2 x$*.*

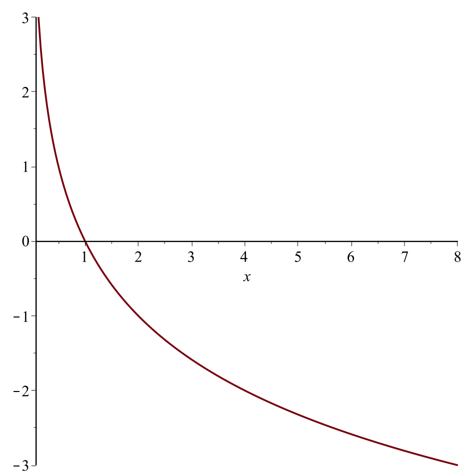

Figure 5.4. *The graph of* $\log_{1/2} x$*.*

Proof. (i) Let $y_1, y_2 > 0$. Set $x_1 = \log_a y_1$, $x_2 = \log_a y_2$, i.e., $a^{x_1} = y_1$ and $y_2 = a^{x_2}$. We have to show that

$$\log_a(y_1 y_2) = x_1 + x_2, \quad \log_a \frac{y_1}{y_2} = x_1 - x_2.$$

We have

$$y_1 y_2 = a^{x_1} a^{x_2} = a^{x_1+x_2} \Rightarrow \log_a(y_1 y_2) = \log_a a^{x_1+x_2} = x_1 + x_2,$$

$$\frac{y_1}{y_2} = \frac{a^{x_1}}{a^{x_2}} = a^{x_1-x_2} \Rightarrow \log_a \frac{y_1}{y_2} = \log_a a^{x_1-x_2} = x_1 - x_2.$$

(ii) Let $x \in \mathbb{R}$ such that $a^x = y$, i.e., $\log_a y = x$. We have to prove that

$$\log_a y^\alpha = \alpha x.$$

We have

$$y^\alpha = (a^x)^\alpha = a^{\alpha x} \Rightarrow \log_a y^\alpha = \log_a a^{\alpha x} = \alpha x.$$

(iii) Let $\beta, x, t \in \mathbb{R}$ such that $a^\beta = b$, $y = a^x = b^t$. Then

$$y = b^t = (a^\beta)^t = a^{t\beta} = a^x.$$

Hence,

$$\log_a y = x = t\beta = (\log_b y)(\log_a b) \Rightarrow \log_b y = \frac{\log_a y}{\log_a b}.$$

(iv) Assume first that $a > 1$. Consider the numbers $y_2 > y_1 > 0$, and set

$$x_1 := \log_a y_1, \ \ x_2 = \log_a y_2.$$

We have to show that $x_2 > x_1$. We argue by contradiction. If $x_1 \geq x_2$, then

$$y_1 = a^{x_1} \geq a^{x_2} = y_2 \Rightarrow y_1 \geq y_2.$$

This contradiction proves the statement (iv) in the case $a > 1$. The case $a \in (0,1)$ is dealt with in a similar fashion.
(v) Assume first that $a > 1$ so that the function $y \mapsto \log_a y$ is increasing. Since $y_n \to y$, we deduce that

$$\frac{y_n}{y} \to 1.$$

Hence, for any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that

$$\forall n > N(\varepsilon) : \frac{y_n}{y_0} \in (a^{-\varepsilon}, a^{\varepsilon}).$$

Hence, $\forall n > N(\varepsilon)$

$$-\varepsilon = \log_a a^{-\varepsilon} < \underbrace{\log_a\left(\frac{y_n}{y_0}\right)}_{=\log_a y_n - \log_a y_0} < \log_a a^{\varepsilon} = \varepsilon \Longleftrightarrow |\log_a y_n - \log_a y_0| < \varepsilon.$$

□

Theorem 5.21. *Fix a real number s and consider $f : (0,\infty) \to (0,\infty)$ given by $f(x) = x^s$. Then for any $c > 0$ and any sequence of positive numbers (x_n) such that $x_n \to c$, we have*

$$\boxed{\lim_{n\to\infty} x_n^s = c^s.}$$

Proof. Set

$$y_n = \log x_n^s = s \log x_n.$$

Theorem 5.20(v) implies that

$$\lim_n y_n = s \lim_n \log x_n = s \log c.$$

Using Theorem 5.18(v), we deduce that

$$\lim_n e^{y_n} = e^{s \log c} = (e^{\log c})^s = c^s.$$

Now observe that

$$e^{y_n} = e^{\log x_n^s} = x_n^s.$$

This proves Theorem 5.21. □

5.3. Limits involving infinities

Suppose that we are given a subset $X \subset \mathbb{R}$ and a function $f : X \to \mathbb{R}$.

Definition 5.22. Let c be a cluster point of X.
(a) We say that the limit of f as $x \to c$ is ∞, and we write this

$$\lim_{x \to c} f(x) = \infty$$

if for any $M > 0$, $\exists \delta = \delta(M) > 0$ such that

$$\forall x \in X \ \ \big(0 < |x - c| < \delta \Rightarrow f(x) > M \big).$$

(b) We say that the limit of f as $x \to c$ is $-\infty$, and we write this

$$\lim_{x \to c} f(x) = -\infty$$

if for any $M > 0$, $\exists \delta = \delta(M) > 0$ such that

$$\forall x \in X \ \ \big(0 < |x - c| < \delta \Rightarrow f(x) < -M \big).$$ □

We have the following version of Proposition 5.3. The proof is left to you.

Proposition 5.23. *Let $f : X \to \mathbb{R}$ be a function defined on a set $X \subset \mathbb{R}$ and c a cluster point of X. Then the following statements are equivalent.*

(i) $\lim_{x \to c} f(x) = \infty$, *i.e., f satisfies (5.1) or (5.2).*

(ii)
$$\forall M > 0, \ \ \exists V \in \mathcal{N}_c^* \ \textit{such that} \ \forall x \in X : x \in V \Rightarrow f(x) \in (M, \infty). \tag{5.13}$$

□

Arguing as in the proof of Theorem 5.4 we obtain the following result. The details are left to you.

Theorem 5.24. *Let c be a cluster point of the set $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ a real valued function on X. The following statements are equivalent.*

(i) $\lim_{x \to c} f(x) = \infty \in \mathbb{R}$.

(ii) *For any sequence $(x_n)_{n \in \mathbb{N}}$ in $X \setminus \{c\}$ such that $x_n \to c$, we have $\lim_n f(x_n) = \infty$.*

□

Observe that if $X \subset \mathbb{R}$ is not bounded above, then for any $M > 0$ the intersection $X \cap (M, \infty)$ is nonempty, i.e., for any number $M > 0$ there exists at least one number $x \in X$ such that $x > M$. Equivalently, this means that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of numbers in X such that

$$\lim_{n \to \infty} x_n = \infty.$$

Definition 5.25. Suppose $X \subset \mathbb{R}$ is a subset not bounded above and $f : X \to \mathbb{R}$ is a real function defined on X.

(a) We say that the limit of f as $x \to \infty$ is the real number A, and we write this $\lim_{x\to\infty} f(x) = A$, if

$$\forall \varepsilon > 0 \;\; \exists M = M(\varepsilon) > 0 \;\; \forall x \in X \;\; \big(x > M \Rightarrow |f(x) - A| < \varepsilon\big).$$

(b) We say that the limit of f as $x \to \infty$ is ∞,and we write this $\lim_{x\to\infty} f(x) = \infty$, if

$$\forall C > 0 \;\; \exists M = M(C) > 0 \;\; \forall x \in X \;\; \big(x > M \Rightarrow f(x) > C\big).$$

(c) We say that the limit of f as $x \to \infty$ is $-\infty$, and we write this $\lim_{x\to\infty} f(x) = -\infty$, if

$$\forall C > 0 \;\; \exists M = M(C) > 0 \;\; \forall x \in X \;\; \big(x > M \Rightarrow f(x) < -C\big). \qquad \square$$

Observe that if $X \subset \mathbb{R}$ is not bounded below, then for any $M > 0$ the intersection $X \cap (-\infty, -M)$ is nonempty, i.e., for any number $M > 0$ there exists at least one number $x \in X$ such that $x < -M$. Equivalently, this means that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of numbers in X such that

$$\lim_{n\to\infty} x_n = -\infty.$$

Definition 5.26. Suppose $X \subset \mathbb{R}$ is a subset not bounded below and $f : X \to \mathbb{R}$ is a real function defined on X.

(a) We say that the limit of f as $x \to -\infty$ is the real number A, and we write this $\lim_{x\to-\infty} f(x) = A$, if

$$\forall \varepsilon > 0 \;\; \exists M = M(\varepsilon) > 0 \;\; \forall x \in X \;\; \big(x < -M \Rightarrow |f(x) - A| < \varepsilon\big).$$

(b) We say that the limit of f as $x \to -\infty$ is ∞, and we write this $\lim_{x\to-\infty} f(x) = \infty$, if

$$\forall C > 0 \;\; \exists M = M(C) > 0 \;\; \forall x \in X \;\; \big(x < -M \Rightarrow f(x) > C\big).$$

(c) We say that the limit of f as $x \to -\infty$ is $-\infty$, and we write this $\lim_{x\to-\infty} f(x) = -\infty$, if

$$\forall C > 0 \;\; \exists M = M(C) > 0 \;\; \forall x \in X \;\; \big(x < -M \Rightarrow f(x) < -C\big). \qquad \square$$

The limits involving infinities have an alternate description involving sequences. Thus, if $X \subset \mathbb{R}$ is not bounded above and $f : X \to \mathbb{R}$ is a real function defined on X, then the equality

$$\lim_{x\to\infty} f(x) = A$$

can be given a characterization as in Theorem 5.4. More precisely, it means that for any sequence of real numbers $x_n \in X$ such that $x_n \to \infty$, the sequence $f(x_n)$ converges to A.

Example 5.27. (a) We want to prove that

$$\boxed{\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e.} \tag{5.14}$$

We will use the fundamental result in Example 4.23 which states that the sequence

$$x_n := \left(1 + \frac{1}{n}\right)^n, \;\; n \in \mathbb{N},$$

converges to the Euler number e. In particular, we deduce that

$$\lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n+1} = e. \tag{5.15}$$

Recall that for any real number x we denote by $\lfloor x \rfloor$ the integer part of the real number x, i.e., the largest integer which is $\leq x$. Thus $\lfloor x \rfloor$ is an integer and

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$$

For $x \geq 1$ we have

$$1 \leq \lfloor x \lfloor \leq x \leq \lfloor x \rfloor + 1$$

and we deduce

$$1 + \frac{1}{\lfloor x \rfloor + 1} \leq 1 + \frac{1}{x} \leq 1 + \frac{1}{\lfloor x \rfloor}.$$

In particular, we deduce that

$$\left(1 + \frac{1}{\lfloor x \rfloor + 1}\right)^{\lfloor x \rfloor} \leq \left(1 + \frac{1}{x}\right)^{\lfloor x \rfloor} \leq \left(1 + \frac{1}{x}\right)^{x} \leq \left(1 + \frac{1}{\lfloor x \rfloor}\right)^{x} \leq \left(1 + \frac{1}{\lfloor x \rfloor}\right)^{\lfloor x \rfloor + 1}. \tag{5.16}$$

From (5.15) we deduce that for any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that

$$\left(1 + \frac{1}{n+1}\right)^{n}, \ \left(1 + \frac{1}{n}\right)^{n+1} \in (e - \varepsilon, e + \varepsilon), \ \ \forall n > N(\varepsilon).$$

If $x > N(\varepsilon) + 1$, then $\lfloor x \rfloor > N(\varepsilon)$ and we deduce from the above that

$$e - \varepsilon < \left(1 + \frac{1}{\lfloor x \rfloor + 1}\right)^{\lfloor x \rfloor} < e + \varepsilon \ \text{ and } \ e - \varepsilon < \left(1 + \frac{1}{\lfloor x \rfloor}\right)^{\lfloor x \rfloor + 1} < e + \varepsilon.$$

The inequalities (5.16) now imply that for $x > N(\varepsilon) + 1$ we have

$$e - \varepsilon < \left(1 + \frac{1}{\lfloor x \rfloor + 1}\right)^{\lfloor x \rfloor} \leq \left(1 + \frac{1}{x}\right)^{x} \leq \left(1 + \frac{1}{\lfloor x \rfloor}\right)^{\lfloor x \rfloor + 1} < e + \varepsilon.$$

This proves (5.14).

(b) We want to prove that

$$\boxed{\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^{x} = e.} \tag{5.17}$$

We will prove that for any sequence of nonzero real numbers (x_n) such that $x_n \to -\infty$, we have

$$\lim_{n} \left(1 + \frac{1}{x_n}\right)^{x_n} = e.$$

Consider the new sequence $y_n := -x_n$. Clearly $y_n \to \infty$. We have

$$\left(1 + \frac{1}{x_n}\right)^{x_n} = \left(1 - \frac{1}{y_n}\right)^{-y_n} = \left(\frac{y_n - 1}{y_n}\right)^{-y_n} = \left(\frac{y_n}{y_n - 1}\right)^{y_n}.$$

Now set $z_n := y_n - 1$ so that $y_n = z_n + 1$ and

$$\left(\frac{y_n}{y_n - 1}\right)^{y_n} = \left(\frac{z_n + 1}{z_n}\right)^{z_n + 1} = \left(1 + \frac{1}{z_n}\right)^{z_n + 1} = \left(1 + \frac{1}{z_n}\right)^{z_n} \times \left(1 + \frac{1}{z_n}\right).$$

Clearly $z_n \to \infty$ so that

$$\lim_{n} \left(1 + \frac{1}{z_n}\right) = 1.$$

Invoking (5.14) we deduce

$$\lim_{n \to \infty} \left(1 + \frac{1}{z_n}\right)^{z_n} = e.$$

Hence,

$$\lim_n \left(1+\frac{1}{x_n}\right)^{x_n} = \lim_n \left(1+\frac{1}{z_n}\right)^{z_n} \times \lim_n \left(1+\frac{1}{z_n}\right) = e.$$

This proves (5.17). □

5.4. One-sided limits

Suppose $X \subset \mathbb{R}$ is a set of real numbers. For any $c \in \mathbb{R}$ we define

$$X_{<c} := \big\{ x \in X;\ \ x < c \big\} = X \cap (-\infty, c),\ \ X_{>c} := \big\{ x \in X;\ \ x > c \big\} = X \cap (c, \infty).$$

Definition 5.28. Let $f : X \subset \mathbb{R}$ and $c \in \mathbb{R}$. We say that L is the left limit of f at c, and we write this

$$L = \lim_{x \nearrow c} f(x) = \lim_{x \to c-} f(x),$$

if

- c is a cluster point of $X_{<c}$ and
- for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\forall x \in X :\ \ x \in (c-\delta, c) \Rightarrow |f(x) - L| < \varepsilon.$$

We say that R is the right limit of f at c, and we write this

$$R = \lim_{x \searrow c} f(x) = \lim_{x \to c+} f(x),$$

if

- c is a cluster point of $X_{>c}$ and
- for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\forall x \in X :\ \ x \in (c, c+\delta) \Rightarrow |f(x) - R| < \varepsilon.$$

□

The next result follows immediately from Theorem 5.4. The details are left to you.

Theorem 5.29. *Let $f : X \to \mathbb{R}$ be a real valued function defined on the set $X \subset \mathbb{R}$. Fix $c \in \mathbb{R}$.*

(a) Suppose that c is a cluster point of $X_{<c}$ and $L \in \mathbb{R}$. Then the following statements are equivalent.

(i)

$$\lim_{x \nearrow c} f(x) = L.$$

(ii) *For any sequence of real numbers (x_n) in X such that $x_n \to c$ and $x_n < c$ $\forall n$ we have*

$$\lim_n f(x_n) = L.$$

(iii) *For any nondecreasing sequence of real numbers (x_n) in X such that $x_n \to c$ and $x_n < c$ $\forall n$ we have*

$$\lim_n f(x_n) = L.$$

(b) Suppose that c is a cluster point of $X_{>c}$ and $L \in \mathbb{R}$. Then the following statements are equivalent.

(i)
$$\lim_{x \searrow c} f(x) = L.$$

(ii) *For any sequence of real numbers (x_n) in X such that $x_n \to c$ and $x_n > c$ $\forall n$ we have*
$$\lim_n f(x_n) = L.$$

(iii) *For any nonincreasing sequence of real numbers (x_n) in X such that $x_n \to c$ and $x_n > c$ $\forall n$ we have*
$$\lim_n f(x_n) = L.$$

□

The next result describes one of the reasons why the one-sided limits are useful. Its proof is left to you as an exercise.

Theorem 5.30. *Consider three real numbers $a < c < b$ and a real valued function $f : (a, b)\setminus\{c\} \to \mathbb{R}$. Then the following statements are equivalent.*

(i)
$$\lim_{x \to c} f(x) = A \in [-\infty, \infty].$$

(ii)
$$\lim_{x \nearrow c} f(x) = \lim_{x \searrow c} f(x) = A \in [-\infty, \infty].$$

□

5.5. Some fundamental limits

In this section we present a collection of examples that play a fundamental role in the development of real analysis.

Example 5.31. We want to prove that
$$\boxed{\lim_{x \to 0} \big(1 + x\big)^{\frac{1}{x}} = e.} \tag{5.18}$$

We invoke Theorem 5.30, so we will prove that
$$\lim_{x \searrow 0} \big(1 + x\big)^{\frac{1}{x}} = \lim_{x \nearrow 0} \big(1 + x\big)^{\frac{1}{x}} = e.$$

We prove first the equality
$$\lim_{x \searrow 0} \big(1 + x\big)^{\frac{1}{x}} = e.$$

We have to prove that if (x_n) is a sequence of *positive* numbers such that $x_n \to 0$, then
$$\lim_n \big(1 + x_n\big)^{\frac{1}{x_n}} = e.$$

Set

$$y_n := \frac{1}{x_n}.$$

Then $y_n \to \infty$ and

$$(1+x_n)^{\frac{1}{x_n}} = \left(1+\frac{1}{y_n}\right)^{y_n},$$

and, according to (5.15), we have

$$\left(1+\frac{1}{y_n}\right)^{y_n} = e.$$

The equality

$$\lim_{x\nearrow 0}(1+x)^{\frac{1}{x}} = e.$$

is proved in a similar fashion invoking (5.17) instead of (5.15). □

Example 5.32. We have ($\log = \log_e$)

$$\boxed{\lim_{x\to 0}\frac{\log(1+x)}{x} = 1.} \tag{5.19}$$

Indeed, consider a sequence of nonzero numbers (x_n) such that $x_n \to 0$. Set

$$y_n = (1+x_n)^{\frac{1}{x_n}}.$$

From (5.19) we deduce that $y_n \to e$. Using Theorem 5.20(v), we deduce that $\log y_n \to \log e = 1$. □

Example 5.33. We have

$$\boxed{\lim_{x\to 0}\frac{e^x-1}{x} = 1.} \tag{5.20}$$

Let $x_n \to 0$. Set $y_n := e^{x_n}$ so that $x_n = \log y_n$ and $y_n \to e^0 = 1$. Next, set $h_n := y_n - 1$ so that $h_n \to 0$. Then

$$\frac{e^{x_n}-1}{x_n} = \frac{y_n-1}{\log y_n} = \frac{h_n}{\log(1+h_n)} = \frac{1}{\frac{\log(1+h_n)}{h_n}} \stackrel{(5.19)}{\longrightarrow} 1.$$ □

Example 5.34. Suppose that $\alpha \in \mathbb{R}$, $\alpha \neq 0$. We have

$$\boxed{\lim_{x\to 0}\frac{(1+x)^\alpha - 1}{x} = \alpha.} \tag{5.21}$$

Let $x_n \to 0$. Then

$$(1+x_n)^\alpha = e^{\alpha\log(1+x_n)}.$$

Set $y_n := \alpha\log(1+x_n)$ so that $y_n \to 0$. Then

$$\frac{(1+x_n)^\alpha - 1}{x_n} = \frac{e^{y_n}-1}{y_n}\cdot\frac{y_n}{x_n} = \frac{e^{y_n}-1}{y_n}\cdot\frac{\alpha\log(1+x_n)}{x_n}.$$

Using (5.20) we deduce

$$\frac{e^{y_n}-1}{y_n} \to 1,$$

and using (5.19) we deduce

$$\frac{\alpha \log(1+x_n)}{x_n} \to \alpha.$$

This shows that

$$\frac{(1+x_n)^\alpha - 1}{x_n} \to \alpha.$$

□

5.6. Trig functions: a less than rigorous definition

Recall that the Cartesian product $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ is called the Cartesian plane and can be visualized as an Euclidean plane equipped with two perpendicular coordinate axes, the x-axis and the y-axis; see Figure 5.5. We can locate a point P in this plane if we can locate its projections P_x and P_y respectively, on the x- and the y-axis respectively; see Figure 5.5. The locations of these projections are indicated by two numbers, the x-coordinate and the y-coordinate respectively, of P. The point with coordinates $(0,0)$ is called the origin and it is denoted by O.

The trigonometric circle is the circle of radius 1 centered at the origin; see Figure 5.5. More precisely, a point with coordinates (x,y) lies on this circle if and only if

$$x^2 + y^2 = 1. \tag{5.22}$$

Additionally, we agree that this circle is given an *orientation*, i.e., a prescribed way of traveling around it. In mathematics, the agreed upon orientation is *counterclockwise orientation* indicated by the arrow along the circle in Figure 5.5.

The starting point of the trigonometric circle is the point S with coordinates $(1,0)$. It can alternatively be described as the intersection of the circle with the positive side of the x-axis. The length[2] of the upper semi-circle is a positive number know by its famous name, π. In particular, the total length of the circle is 2π.

Suppose that we start at the point S and we travel along the circle, in the counterclockwise direction a distance $\theta \geq 0$. We denote by P the final point of this journey. The coordinates of this point depend only on the distance θ traveled. The x-coordinate of P is denoted by $\cos\theta$, and the y-coordinate of P is denoted by $\sin\theta$. The equality (5.22) implies that

$$\cos^2\theta + \sin^2\theta = 1, \;\; \forall \theta \geq 0. \tag{5.23}$$

Observe that if we continue our journey from P in the counterclockwise direction for a distance 2π then we are back at P. This shows that

$$\cos(\theta + 2\pi) = \cos\theta, \;\; \sin(\theta+2\pi) = \sin\theta, \;\; \forall\theta \geq 0. \tag{5.24}$$

We can define $\cos\theta$ and $\sin\theta$ for negative θ's as well. Suppose that $\theta = -\phi$, $\phi \geq 0$. If we start at S and travel along the circle in the *clockwise* direction a distance ϕ, then we reach a point Q. By definition, its coordinates are $\cos(-\phi)$ and $\sin(-\phi)$; see Figure 5.6.

From the description it is easily seen that

$$\cos(-\phi) = \cos\phi, \;\; \sin(-\phi) = -\sin\phi, \;\; \forall\phi \geq 0. \tag{5.25}$$

[2] We avoid explaining what length means.

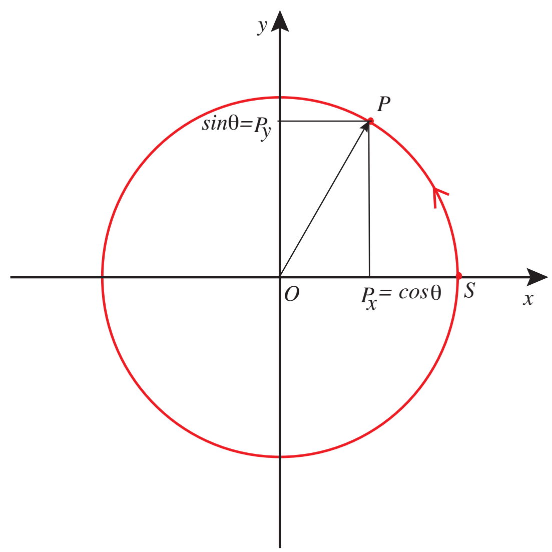

Figure 5.5. *The trigonometric circle. The distance of the journey from S to P in the counterclockwise direction is θ.*

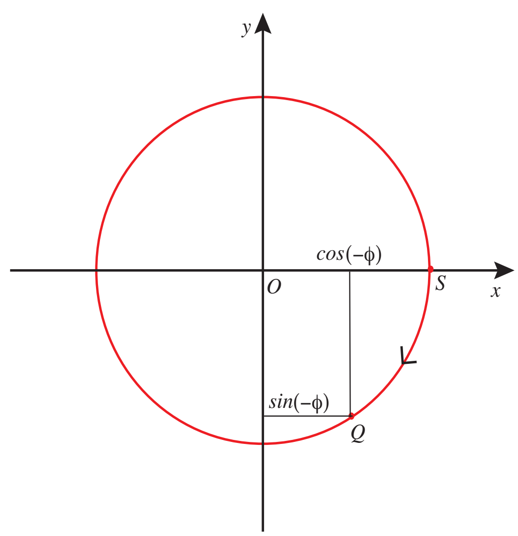

Figure 5.6. *The trigonometric circle. The distance of the journey from S to Q in the* ***clockwise*** *direction is ϕ.*

We have thus constructed two functions

$$\cos, \sin : \mathbb{R} \to \mathbb{R},$$

called *trigonometric functions*. Their graphs are depicted in Figure 5.7 and 5.8.

Let us record a few important values of these functions.

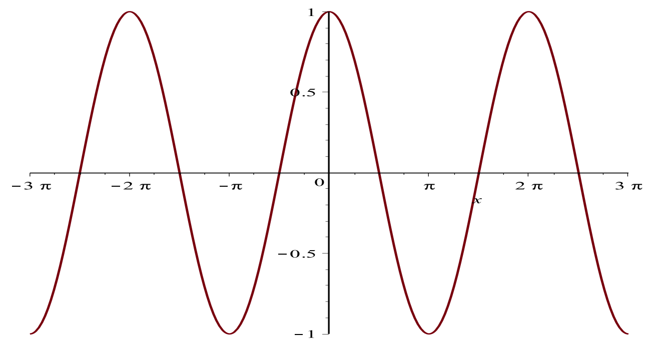

Figure 5.7. *The graph of* $\cos x$

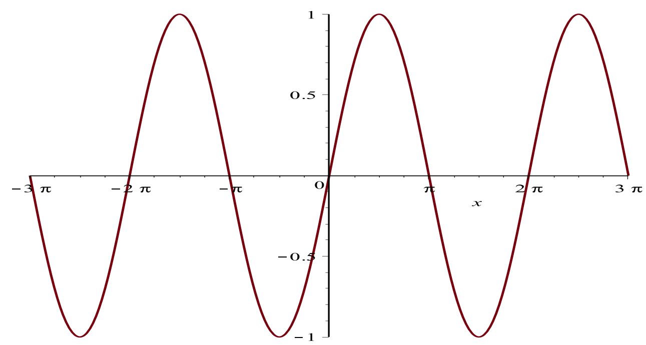

Figure 5.8. *The graph of* $\sin x$

Table 5.1. Some important values of trig functions

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	0

We list below some of the more elementary, but very important, properties of the trigonometric functions sin and cos.

$$\cos^2 x + \sin^2 x = 1, \ \ \forall x \in \mathbb{R}. \tag{5.26a}$$

$$\cos(x + 2\pi) = \cos x, \ \ \sin(x + 2\pi) = \sin x, \ \ \forall x \in \mathbb{R}. \tag{5.26b}$$

$$\cos(-x) = \cos x, \ \ \sin(-x) = -\sin(x), \ \ \forall x \in \mathbb{R}. \tag{5.26c}$$

$$\cos(x + \pi) = -\cos(x), \ \ \sin(x + \pi) = -\sin(x), \ \ \forall x \in \mathbb{R}, \tag{5.26d}$$

$$\sin\left(x + \frac{\pi}{2}\right) = \cos x, \ \ \forall x \in \mathbb{R}. \tag{5.26e}$$

$$|\cos x| \le 1, \ \ |\sin x| \le 1, \ \ \forall x \in \mathbb{R}. \tag{5.26f}$$

$$\cos x > 0, \ \ \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \ \text{ and } \ \sin x > 0, \ \ \forall x \in (0, \pi). \tag{5.26g}$$

$$\cos x = 0 \Longleftrightarrow x \text{ is an odd multiple of } \tfrac{\pi}{2}, \ \ \sin x = 0 \Longleftrightarrow x \text{ is a multiple of } \pi. \tag{5.26h}$$

Definition 5.35. Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function defined on the real axis $\mathbb{R}$.

(i) The function f is called *even* if

$$f(-x) = f(x), \ \ \forall x \in \mathbb{R}.$$

(ii) The function f is called *odd* if

$$f(-x) = -f(x), \ \ \forall x \in \mathbb{R}.$$

(iii) Suppose P is a positive real number. We say that f is P-*periodic* if

$$f(x+P) = f(x), \ \ \forall x \in \mathbb{R}.$$

(iv) The function f is called periodic if there exists $P > 0$ such that f is P-periodic. Such a number P is called a *period* of f.

□

We see that the functions $\cos x$ and $\sin x$ are 2π-periodic, $\cos x$ is even, and $\sin x$ is odd.

In applications, we often rely on other trigonometric functions derived from sin and cos. We define

$$\tan x = \frac{\sin x}{\cos x}, \ \ \text{whenever } \cos x \neq 0,$$
$$\cot x = \frac{\cos x}{\sin x}, \ \ \text{whenever } \sin x \neq 0.$$

The graphs of $\tan x$ and $\cot x$ are depicted in Figure 5.9 and 5.10.

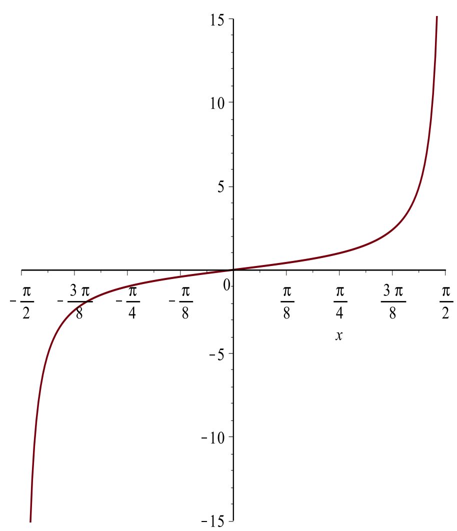

Figure 5.9. *The graph of* $\tan x$ *for* $x \in (-\pi/2, \pi/2)$

Example 5.36. We want to outline a geometric explanation for an important limit.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.} \tag{5.27}$$

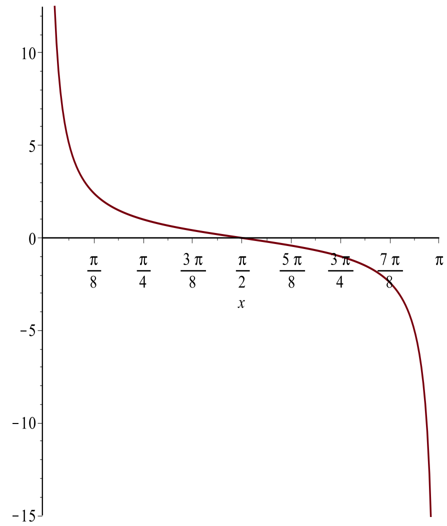

Figure 5.10. *The graph of* $\cot x$ *for* $x \in (0, \pi)$

We will prove that

$$\lim_{x \nearrow 0} \frac{\sin x}{x} = \lim_{x \searrow 0} \frac{\sin x}{x} = 1.$$

Since

$$\frac{\sin x}{x} = \frac{\sin(-x)}{-x}$$

it suffices to prove only that

$$\lim_{x \searrow 0} \frac{\sin x}{x} = 0. \tag{5.28}$$

This will follow immediately from the following fundamental inequalities

$$\theta \cos^2 \theta \leq \sin \theta \leq \theta, \;\; \forall 0 < \theta < \frac{\pi}{2}. \tag{5.29}$$

Let us temporarily take for granted these inequalities and how how they imply (5.28).

Observe that (5.29) implies that

$$0 \leq \sin \theta \leq \theta, \;\; \forall 0 < \theta < \frac{\pi}{2}.$$

The Squeezing Principle shows that

$$\lim_{\theta \searrow 0} \sin \theta = 0. \tag{5.30}$$

This shows that the limit

$$\lim_{\theta \searrow 0} \frac{\sin \theta}{\theta}$$

is a bad limit of the type $\frac{0}{0}$.

We can rewrite (5.29) as

$$1 - \sin^2 \theta = \cos^2 \theta \leq \frac{\sin \theta}{\theta} \leq 1. \tag{5.31}$$

The equality (5.30) shows that

$$\lim_{\theta\searrow 0}(1-\sin^2\theta)=1.$$

The equality (5.29) now follows by applying the Squeezing Principle to the inequalities (5.31).

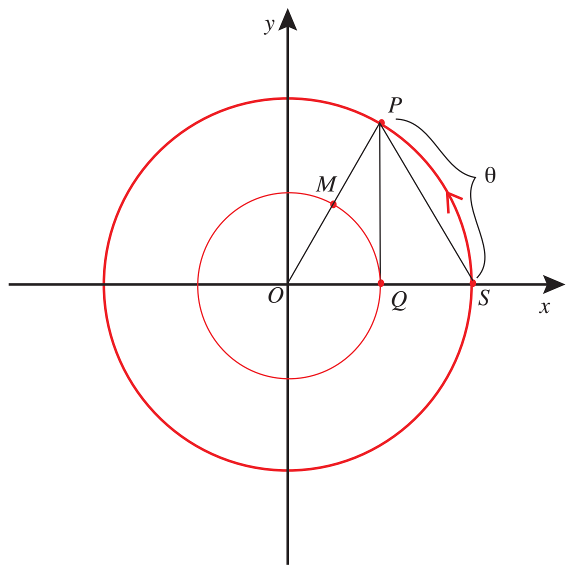

Figure 5.11. *The trigonometric circle. The distance of the journey from S to P in the* ***counterclockwise*** *direction is θ.*

"Proof" of (5.29). Fix θ, $0<\theta<\frac{\pi}{2}$. We denote by P the point on the trigonometric circle reached from S by traveling a distance θ in the counterclockwise direction; see Figure 5.11. Denote by Q the projection of P onto the x-axis. We have

$$|OQ|=\cos\theta,\ \ |PQ|=\sin\theta.$$

Denote by M the intersection of the line OP with the circle centered at O and radius $|OP|=\cos\theta$. We distinguish three regions in Figure 5.12: the circular sector (OQM), the triangle $\triangle OSP$, and the circular sector (OSP). Clearly

$$(OQM)\subset\triangle OSP\subset(OSP)$$

so that we obtain inequalities between their areas[3]

$$\operatorname{area}(OQM)\leq\operatorname{area}\triangle OSP\leq\operatorname{area}(OSP)$$

The area of a circular sector is[4]

$$\frac{1}{2}\times\text{square of the radius of the sector}\times\text{the size of the angle of the sector}.\tag{5.32}$$

We have

$$\operatorname{area}(OQM)=\frac{1}{2}|OQ|^2\theta=\frac{\theta\cos^2\theta}{2},\ \ \operatorname{area}(OSP)=\frac{1}{2}|OS|^2\theta=\frac{\theta}{2},$$

$$\operatorname{area}\triangle OSP=\frac{1}{2}|PQ|\times|OS|=\frac{1}{2}\sin\theta.$$

Hence,

$$\frac{\theta\cos^2\theta}{2}\leq\frac{1}{2}\sin\theta\leq\frac{\theta}{2}.$$ □

[3]At this point we do not have a rigorous definition of the area of a planar region.

[4]The equality (5.32) needs a justification

5.7. Useful trig identities.

We list here a few important trigonometric identities that we will use in the future.

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x, \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y). \tag{5.33a}$$

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x. \tag{5.33b}$$

$$\frac{1 + \cos x}{2} = \cos^2(x/2), \quad \frac{1 - \cos x}{2} = \sin^2(x/2). \tag{5.33c}$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big), \quad \sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big). \tag{5.33d}$$

$$\tan(x \pm y) = \frac{\tan x + \tan y}{1 \mp \tan x \tan y}. \tag{5.33e}$$

5.8. Landau's symbols

Let $c \in [-\infty, \infty]$ and consider two real valued functions f, g defined on the same set X which admits c as a cluster point. We say that

$$f(x) = O\big(g(x)\big) \text{ as } x \to c \tag{5.34}$$

if there exists a positive constant C and a neighborhood U of c such that

$$\forall x \in X, \quad x \in (X \cap U) \setminus \{c\} \Rightarrow |f(x)| \leq C|g(x)|.$$

For example

$$\frac{x}{x^2+1} = O\left(\frac{1}{x}\right) \text{ as } x \to \infty.$$

We say that

$$f(x) = o\big(g(x)\big) \text{ as } x \to c, \tag{5.35}$$

if for any $\varepsilon > 0$ there exists a neighborhood U_ε of c such that

$$\forall x \in X, \quad x \in U_\varepsilon \setminus \{c\} \Rightarrow |f(x)| \leq \varepsilon |g(x)|.$$

If $g(x) \neq 0$ for any x in a neighborhood U of c, then

$$f(x) = o\big(g(x)\big) \text{ as } x \to c \iff \lim_{x \to c} \frac{f(x)}{g(x)} = 0.$$

Loosely speaking, this means that $f(x)$ is much, much smaller than $g(x)$ as x approaches c. For example

$$e^{-x} = o(x^{-25}) \text{ as } x \to \infty,$$

and

$$x^3 = o(x^2) \text{ as } x \to 0.$$

Finally, we say that f is *similar* to $g(x)$ as $x \to c$, and we write this

$$f(x) \sim g(x) \text{ as } x \to c$$

if

$$\lim_{x \to c} \frac{f(x)}{g(x)} = 1.$$

For example

$$x^3 - 39x^2 + 17 \sim x^3 + 3x^2 + 2x + 1 \text{ as } x \to \infty,$$

and

$$e^x - 1 \sim x \ \text{ as } \ x \to 0.$$

5.9. Exercises

Exercise 5.1. Prove that any real number is a cluster point of the set of rational numbers.□

Exercise 5.2. Prove Proposition 5.3. □

Exercise 5.3 (Squeezing principle)**.** Let $f, g, h : X \to \mathbb{R}$ be three functions defined on the same subset $X \subset \mathbb{R}$ and c a cluster point of X. Suppose that U is a deleted neighborhood of c such that

$$f(x) \leq h(x) \leq g(x), \;\; \forall x \in U \cap X.$$

Show that if

$$\lim_{x \to c} f(x) = A = \lim_{x \to c} g(x),$$

then

$$\lim_{x \to c} h(x) = A.$$

□

Exercise 5.4. Consider a subset $X \subset \mathbb{R}$, a function $f : X \to \mathbb{R}$, and a cluster point c of the set X. Prove that the following statements are equivalent.

(i) The limit $\lim_{x \to c} f(x)$ exists and it is finite.

(ii) For any sequence $(x_n)_{n \in \mathbb{N}} \subset X$ such that $x_n \to c$ and $x_n \neq c$, $\forall n \in \mathbb{N}$ the sequence $\big(f(x_n) \big)_{n \in \mathbb{N}}$ is convergent.

□

Exercise 5.5. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a function. Suppose that f is a *Lipschitz function*, i.e., there exists a constant L such that

$$|f(x) - f(y)| \leq L|x - y|, \;\; \forall x, y \in I.$$

Show that for any $y \in Y$ we have

$$\lim_{x \to y} f(x) = f(y).$$

□

Exercise 5.6. We already know that the series

$$\sum_{n \geq 1} \frac{1}{n^s}$$

converges for any *rational number* $s > 1$. Prove that it converges for any *real number* $s > 1$.□

Exercise 5.7. (a) Prove that for any $n \in \mathbb{N}$ we have

$$\lim_{x \to \infty} \frac{1}{x^n} = 0.$$

(b)Let $k \in \mathbb{N}$ and consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x^{2k}}$. Show that

$$\lim_{x \to 0} f(x) = \infty.$$

□

Exercise 5.8. Fix a natural number n. Consider the polynomial

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Show that

$$\lim_{x\to\infty} P(x) = \infty, \quad \lim_{x\to-\infty} P(x) = \begin{cases} \infty, & n \text{ is even} \\ -\infty & , n \text{ is odd.} \end{cases}$$ □

Exercise 5.9. Consider two convergent sequences of real numbers $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$. We set

$$x := \lim_{n\to\infty} x_n, \quad y := \lim_{n\to\infty} y_n.$$

Show that if $x_n > 0$, $\forall n \geq 0$ and $x > 0$ then

$$\lim_{n\to\infty} x_n^{y_n} = x^y.$$

Hint. Use the same strategy as in the proof of Theorem 5.21. □

Exercise 5.10. Prove that

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x, \quad \forall x \in \mathbb{R}.$$

Hint: Use the result in Example 5.27 and Theorem 5.21. □

Exercise 5.11. Fix an arbitrary number $a > 1$.

(a) Prove that for any $x > 1$ we have

$$a^x \geq a^{\lfloor x \rfloor} \geq 1 + (a-1)\lfloor x \rfloor + \binom{\lfloor x \rfloor}{2}(a-1)^2.$$

(b) Prove that

$$\lim_{x\to\infty} \frac{x}{a^x} = 0, \quad \lim_{x\to\infty} \frac{a^x}{x} = \infty.$$

Hint. Use (a) and Example 4.18.

(c) Let $r > 0$. Prove that

$$\lim_{x\to\infty} \frac{x^r}{a^x} = 0.$$

Hint. Reduce to (b).

(d) Prove that

$$\lim_{x\to\infty} \frac{\log_a x}{x} = 0.$$

Hint. Reduce to (c). □

Exercise 5.12. Fix a positive real number s and consider the function $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^s$.

(a) Show that f is an increasing function.

(b) Show that

$$\lim_{x\searrow 0} x^s = 0.$$

Exercise 5.13. Let $a, b \in \mathbb{R}$, $a < b$. Prove that if $f : (a,b) \to \mathbb{R}$ is a nondecreasing function and $x_0 \in (a,b)$, then the one sided limits

$$\lim_{x\nearrow x_0} f(x) \ \text{ and } \ \lim_{x\searrow x_0} f(x)$$

exist and

$$\lim_{x \nearrow x_0} f(x) = \sup_{x < x_0} f(x), \quad \lim_{x \searrow x_0} f(x) = \inf_{x > x_0} f(x). \qquad \square$$

Exercise 5.14. Consider the function

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f(x) = x \sin\left(\frac{1}{x}\right).$$

Prove that

$$\lim_{x \to 0} f(x) = 0. \qquad \square$$

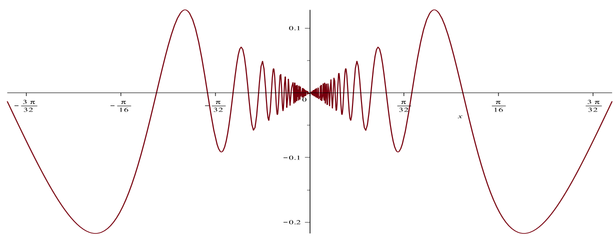

Figure 5.12. *The graph of* $x \sin(1/x)$ *for* $|x| < \pi/10$.

Exercise 5.15. Consider $f : [0, \infty) \to \mathbb{R}$,

$$f(x) = \frac{2x^3 + x^2}{x^3 + x^2 + 1}.$$

Show that

$$f(x) = O(1) \ \text{ as } x \to \infty.$$

Above, we used Landau's notation introduced in section 5.8. $\square$

Exercise 5.16. (a) Prove Lemma 5.15.

Hint. The case $a = 1$ is trivial. In the case $a > 1$ show that there exists a sequence of positive rational numbers (r_n) such that

$$-r_n \le x_n - x \le r_n, \quad \forall n.$$

Now use Lemma 5.9, Lemma 5.12, and the Squeezing Principle to conclude. The case $a < 1$ follows from the case $a > 1$.

(b) Prove the equality (5.11).

Hint. First prove that (5.11) holds for any $x \in \mathbb{Q}$. Then conclude using Lemma 5.15. $\square$

Exercise 5.17. Prove Proposition 5.23. $\square$

Exercise 5.18. Prove Theorem 5.24. $\square$

Exercise 5.19. Prove Theorem 5.29. $\square$

Exercise 5.20. Prove Theorem 5.30. $\square$

5.10. Exercises for extra credit

Exercise* 5.1 (Viète)**.** Consider the sequence $(x_n)_{n\geq 0}$ defined by

$$x_0 = 0, \;\; x_{n+1} = \sqrt{\frac{1+x_n}{2}}, \;\; \forall n \geq 0.$$

(a) Prove that

$$x_n = \cos\frac{\pi}{2^{n+1}}, \;\; \forall n \geq 0.$$

(b) Prove that

$$\lim_{n\to\infty}\left(x_1 \cdot x_2 \cdots x_n\right) = \frac{2}{\pi}. \qquad \square$$

Chapter 6

Continuity

6.1. Definition and examples

The concept of continuity is a fundamental mathematical concept with wide a range of applications.

Definition 6.1. Suppose that $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ is a real valued function defined on X. We say that the function f is *continuous at a point* $x_0 \in X$ if

$$\forall \varepsilon > 0 \;\; \exists \delta = \delta(\varepsilon) > 0 \;\text{ such that }\; \forall x \in X \;\; |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

We say that the function f is *continuous (on X)* if it is continuous at every point $x_0 \in X$.□

Arguing as in the proof of Theorem 5.4 we obtain the following very useful alternate characterization of continuity. The details are left to you as an exercise.

Theorem 6.2. *Let $X \subset \mathbb{R}$, $x_0 \in X$, and $f : X \to \mathbb{R}$ a real valued function on X. The following statements are equivalent.*

(i) *The function f is continuous at x_0.*

(ii) *For any sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $x_n \to x_0$, we have $\lim_n f(x_n) = f(x_0)$.*

□

We have the following useful consequence which relates the concept of continuity to the concept of limit. Its proof is left to you as an exercise.

Corollary 6.3. *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. Suppose that $x_0 \in X$ is a cluster point of X. Then the following statements are equivalent.*

(i) *The function f is continuous at x_0.*

(ii) $\lim_{x \to x_0} f(x) = f(x_0)$.

□

We have already encountered many examples of continuous functions.

Example 6.4. (a) Let $k \in \mathbb{N}$. Then the function

$$f : \mathbb{R} \to \mathbb{R}, \;\; f(x) = x^k, \;\; \forall x \in \mathbb{R},$$

is continuous on its domain $\mathbb{R}$. Indeed, if $x_0 \in \mathbb{R}$ and $(x_n)_{n \in \mathbb{R}}$ is a sequence of real numbers such that $x_n \to x_0$, then Proposition 4.15 implies that

$$x_n^k \to x_0^k,$$

thus proving the continuity of f at an arbitrary point $x_0 \in \mathbb{R}$.

(b) A similar argument shows that if $k \in \mathbb{N}$, then the function

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \;\; f(x) = \frac{1}{x^k}, \;\; \forall x \in \mathbb{R} \setminus \{0\},$$

is continuous.

(c) Fix $s \in \mathbb{R}$. Then the function

$$f : (0, \infty) \to \mathbb{R}, \;\; f(x) = x^s, \;\; \forall x > 0,$$

is continuous. Indeed, this follows by invoking Theorems 5.21 and 6.2.

(d) Let $a > 0$. Then the functions

$$f : \mathbb{R} \to (0, \infty), \;\; f(x) = a^x,$$

and

$$g : (0, \infty) \to \mathbb{R}, \;\; g(x) = \log_a x,$$

are continuous on their domains. Indeed, the continuity of f follows from Lemma 5.15, while the continuity of g follows from Theorem 5.20.

(e) The trigonometric functions

$$\sin, \cos : \mathbb{R} \to \mathbb{R}$$

are continuous.

Let us first prove that these functions are continuous at $x_0 = 0$. The continuity of sin at $x_0 = 0$ follows immediately from (5.30) and Corollary 6.3. To prove the continuity of cos at $x_0 = 0$ we have to show that if (x_n) is a sequence of real numbers such that $x_n \to 0$, then $\cos x_n \to \cos 0 = 1$.

Let (x_n) be a sequence of real numbers converging to zero. Then

$$\cos^2 x_n = 1 - \sin^2 x_n$$

and we deduce that

$$\lim_n \cos^2 x_n = 1 - \sin^2 x_n = \lim_n (1 - \sin^2 x_n) = 1.$$

Since $x_n \to 0$, we deduce that there exists $N_0 > 0$ such that $|x_n| < \frac{\pi}{2}$, $\forall n > N_0$. The inequalities (5.26g) imply that

$$\cos x_n > 0, \;\; \forall n >_0,$$

so that,

$$\cos x_n = \sqrt{1 - \sin^2 x_n}, \;\; \forall n > N_0.$$

Exercise 4.15 now implies that

$$\lim_n \cos x_n = \sqrt{\lim_n (1 - \sin^2 x_n)} = \sqrt{1} = \cos 0.$$

We can now prove the continuity of sin and cos at an arbitrary point x_0. Suppose that x_n is a sequence of real numbers such that $x_n \to x_0$. We have to show that

$$\lim_n \sin x_n = \sin x_0 \;\; \text{and} \;\; \lim_n \cos x_n = \cos x_0.$$

We set $h_n = x_n - x_0$, so that, $x_n = x_0 + h$. Then

$$\sin x_n = \sin(x_0 + h_n) \overset{(5.33a)}{=} \sin x_0 \cos h_n + \sin h_n \cos x_0$$

and

$$\cos x_n = \cos(x_0 + h_n) \overset{(5.33a)}{=} \cos x_0 \cos h_n - \sin x_0 \sin h_n.$$

Observe that $h_n \to 0$ and, since sin and cos are continuous at 0, we have $\sin h_n \to 0$ and $\cos h_n \to 1$. We deduce

$$\lim_n \sin x_n = \lim_n \sin(x_0 + h_n) = \sin x_0 \lim_n \cos h_n + \cos x_0 \lim_n \sin h_n = \sin x_0$$

and

$$\lim_n \cos x_n = \lim_n \cos(x_0 + h_n) = \cos x_0 \lim_n \cos h_n - \sin x_0 \lim_n \sin h_n = \cos x_0.$$

(f) Recall that a function $f : X \to \mathbb{R}$, $X \subset \mathbb{R}$ is called *Lipschitz* if

$$\exists L > 0 : \ \ \forall x_1, x_2 \in X \ \ |f(x_1) - f(x_2)| \leq L|x_1 - x_2|.$$

Observe that a Lipschitz function is necessarily continuous. Indeed, if $x_0 \in X$ and (x_n) is a sequence in X such that $x_n \to x_0$ then

$$|f(x_n) - f(x_0)| \leq L|x_n - x_0| \to 0,$$

and the squeezing principle implies that $f(x_n) \to f(x_0)$.

Observe that the absolute value function $f : \mathbb{R} \to [0, \infty)$, $f(x) = |x|$ is Lipschitz because of the following elementary inequality (see Exercise 4.5)

$$|f(x) - f(y)| = \big|\, |x| - |y| \,\big| \leq |x - y|, \ \ \forall x, y \in \mathbb{R}. \tag{6.1}$$

Thus the absolute value function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is a continuous function. □

Proposition 6.5. *Let $X \subset \mathbb{R}$, $c \in \mathbb{R}$, and suppose that $f, g : X \to \mathbb{R}$ are two continuous functions. Then the functions*

$$f + g, cf, f \cdot g : X \to \mathbb{R},$$

are continuous. Additionally, if $\forall x \in X \ g(x) \neq 0$, then the function

$$\frac{f}{g} : X \to \mathbb{R}$$

is also continuous.

Proof. This is an immediate consequence of Proposition 4.15 and Theorem 6.2. □

Example 6.6. Polynomial function $p : \mathbb{R} \to \mathbb{R}$ defined by

$$p(x) = c_n x^n + \cdots + c_1 x + c_0,$$

$n \in \mathbb{Z}$, $n \geq 0$, $c_0, \dots, c_n \in \mathbb{R}$ are continuous. For example, the function $p(x) = x^3 - 2x + 5$, $x \in \mathbb{R}$, is continuous on $\mathbb{R}$.

We can easily get more complicated examples. Thus the function $(x^3 - 2x + 5)\sin x$, $x \in \mathbb{R}$, is continuous, the function $e^x + e^{-x}$, $x \in \mathbb{R}$, is continuous and nowhere zero, so the quotient

$$\frac{(x^3 - 2x + 5)\sin x}{e^x + e^{-x}}, \ \ x \in \mathbb{R}$$

is also continuous on $\mathbb{R}$. □

Proposition 6.7. *Suppose that $X, Y \subset \mathbb{R}$ and that $f: X \to \mathbb{R}$ and $g: Y \to \mathbb{R}$ are continuous functions such that*

$$f(X) \subset Y.$$

Then the composition $g \circ f: X \to \mathbb{R}$, $g \circ f(x) = g(f(x))$, $\forall x \in X$ is also a continuous function.

Proof. Theorem 6.2 implies that we have to prove that for any $x_0 \in X$ and any sequence (x_n) in X such that $x_n \to x_0$ we have

$$g(f(x_n)) \to g(f(x_0)).$$

Set $y_0 := f(x_0)$, $y_n := f(x_n)$. Since f is continuous at x_0, Theorem 6.2 shows that $f(x_n) \to f(x_0)$, i.e., $y_n \to y_0$. Since g is continuous at y_0, Theorem 6.2 implies that $g(y_n) \to g(y_0)$, i.e.,

$$g(f(x_n)) \to g(f(x_0)).$$

□

Example 6.8. Consider the continuous functions

$$f, g, h: \mathbb{R} \to \mathbb{R}, \quad f(x) = \sin x, \quad g(x) = e^x, \quad h(x) = |x|.$$

Then $g \circ f(x) = e^{\sin x}$ is continuous on $\mathbb{R}$, and so is the function $f \circ g(x) = \sin e^x$. Similarly $f \circ h(x) = \sin|x|$ is a continuous function on $\mathbb{R}$. □

Definition 6.9. Let $X \subset \mathbb{R}$ be a set of real numbers and $f: X \to \mathbb{R}$ a real valued function on X.

(a) The sequence of functions $f_n: X \to \mathbb{R}$, $n \in \mathbb{N}$ is said to converge *pointwisely* to the function $f: X \to \mathbb{R}$ if

$$\lim_{n\to\infty} f_n(x) = f(x), \quad \forall x \in X,$$

i.e.,

$$\forall \varepsilon > 0, \ \forall x \in X \ \exists N = N(\varepsilon, x): \ \forall n > N(\varepsilon, x) \ |f_n(x) - f(x)| < \varepsilon. \tag{6.2}$$

(b) The sequence of functions $f_n: X \to \mathbb{R}$, $n \in \mathbb{N}$ is said to *converge uniformly* to the function $f: X \to \mathbb{R}$ if

$$\forall \varepsilon > 0 \ \exists N = N(\varepsilon) > 0 \text{ such that } \forall n > N(\varepsilon), \ \forall x \in X: \ |f_n(x) - f(x)| < \varepsilon. \tag{6.3}$$

□

Theorem 6.10 (Continuity of uniform limits)**.** *Let $X \subset \mathbb{R}$ be a set of real numbers. If the sequence of* continuous *functions*

$$f_n: X \to \mathbb{R}, \quad n \in \mathbb{N}$$

converges uniformly to the function $f: X \to \mathbb{R}$, then the limit function f is also continuous on X.

Proof. We have to prove that given $x_0 \in X$ the function f is continuous at x_0, i.e., we have to show that

$$\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0 \ \forall x \in X \ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \tag{6.4}$$

Let $\varepsilon > 0$. The uniform convergence implies that

$$\exists N(\varepsilon) > 0 : \quad \forall x \in X, \quad \forall n > N(\varepsilon) \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3}. \tag{6.5}$$

Fix $n_0 > N(\varepsilon)$. The function f_{n_0} is continuous at x_0 and thus

$$\exists \delta(\varepsilon) > 0 \ \ \forall x \in X : \quad |x - x_0| < \delta(\varepsilon) \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3}. \tag{6.6}$$

We deduce that if $|x - x_0| < \delta(\varepsilon)$, then

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)|. \tag{6.7}$$

From (6.5) we deduce that since $n_0 > N(\varepsilon)$ we have

$$|f(x) - f_{n_0}(x)|, \ \ |f_{n_0}(x_0) - f(x_0)| < \frac{\varepsilon}{3}, \ \ \forall x \in X.$$

From (6.6) we deduce that if $|x - x_0| < \delta(\varepsilon)$, then

$$|f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3}.$$

Using these facts in (6.7) we deduce that if $|x - x_0| < \delta(\varepsilon)$, then

$$|f(x) - f(x_0)| < \varepsilon.$$

$\square$

6.2. Fundamental properties of continuous functions

In this section we will discuss several fundamental properties of continuous functions, which hopefully will explain the usefulness of the concept of continuity.

Theorem 6.11. *Suppose that c is an arbitrary real number, $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function continuous at $x_0 \in X$.*

(a) If $x_0 \in X$ satisfies $f(x_0) < c$, then there exists $\delta > 0$ such that

$$\forall x \in X \ \ (\,|x - x_0| < \delta \Rightarrow f(x) < c\,).$$

In other words, if $f(x_0) < c$, then for any $x \in X$ sufficiently close to x_0 we also have $f(x) < c$.

(b) If $x_0 \in X$ satisfies $f(x_0) > c$, then there exists $\delta > 0$ such that

$$\forall x \in X \ \ (\,|x - x_0| < \delta \Rightarrow f(x) > c\,).$$

In other words, if $f(x_0) > c$, then for any $x \in X$ sufficiently close to x_0 we also have $f(x) > c$.

Proof. Fix $\varepsilon_0 > 0$, such that $f(x_0) + \varepsilon_0 < c$. (For example, we can choose $\varepsilon_0 = \frac{1}{2}(c - f(x_0)\,)$.)

The continuity of f at x_0 (Definition 6.1) implies that there exists $\delta_0 > 0$ such that for any $x \in X$ satisfying $|x - x_0| < \delta_0$ we have

$$|f(x) - f(x_0)| < \varepsilon_0,$$

so that

$$f(x_0) - \varepsilon_0 < f(x) < f(x_0) + \varepsilon_0 < c.$$

$\square$

Corollary 6.12. *Suppose that $X \subset \mathbb{R}$, $x_0 \in X$ and $f : X \to \mathbb{R}$ is a continuous function such that $f(x_0) \neq 0$. Then there exists $\delta > 0$ such that*

$$\forall x \in X \ \ (\, |x - x_0| < \delta \Rightarrow f(x) \neq 0\,).$$

In other words, if $f(x_0) \neq 0$, then for any $x \in X$ sufficiently close to x_0 we also have $f(x) \neq 0$.

Proof. Consider the function $g : X \to \mathbb{R}$, $g(x) = |f(x)|$. The function g is continuous because it is the composition of the absolute-value-function with the continuous function f. Additionally $|g(x_0)| > 0$. The desired conclusion now follows from Theorem 6.11 (b). □

To state and prove our next result we need to make a small digression. Recall that the *Completeness Axiom* states that if the set $X \subset \mathbb{R}$ is *bounded above*, then it admits a least upper bound which is a *real number* denoted by $\sup X$. If the set X is not bounded above, then we define $\sup X := \infty$. Thus we gave a meaning to $\sup X$ *for any* subset $X \subset \mathbb{R}$. Moreover,

$$\sup X < \infty \Longleftrightarrow \text{the set } X \text{ is bounded above.}$$

Similarly, we define $\inf X = -\infty$ for any set X that is not bounded below. Thus we gave a meaning to $\inf X$ *for any* subset $X \subset \mathbb{R}$. Moreover,

$$\inf X > -\infty \Longleftrightarrow \text{the set } X \text{ is bounded below.}$$

Lemma 6.13. *(a) If Y is a set of real numbers and $M = \sup Y \in (-\infty, \infty]$, then there exists an increasing sequence of real numbers $(M_n)_{n \geq 1}$ and a sequence (y_n) in Y such that*

$$M_n \leq y_n \leq M,\ \ \forall n,\ \ \lim_n M_n = M.$$

(b) If Y is a set of real numbers and $m = \inf Y \in [-\infty, \infty)$, then there exists a decreasing sequence of real numbers $(m_n)_{n \geq 1}$ and a sequence (y_n) in Y such that

$$m \leq y_n \leq m_n,\ \ \forall n,\ \ \lim_n m_n = m.$$

Proof. We prove only (a). The proof of (b) is very similar and it is left to you as an exercise. We distinguish two cases.

A. $M < \infty$. Since M is the least upper bound of Y, for any $n > 0$ there exists $y_n \in X$ such that

$$M - \frac{1}{n} \leq y_n \leq M.$$

The sequences (y_n) and $M_n = M - \frac{1}{n}$ have the desired properties.

B. $M = \infty$. Hence, the set Y is not bounded above. Thus, for any $n \in \mathbb{N}$ there exists $y_n \in Y$ such that $y_n \geq n$. The sequences (y_n) and $M_n = n$ have the desired properties. □

Theorem 6.14 (Weierstrass)**.** *Consider a continuous function f defined on a* ***closed and bounded*** *interval $f : [a, b] \to \mathbb{R}$. Then the following hold.*

(i)
$$M := \sup\{\, f(x);\ \ x \in [a, b]\,\} < \infty.$$

(ii) $\exists x^* \in [a, b]$ *such that $f(x^*) = M$.*

(iii)
$$m := \inf\left\{ f(x);\ \ x \in [a,b] \right\} > -\infty.$$
(iv) $\exists x_* \in [a,b]$ *such that* $f(x_*) = m$.

Proof. We prove only (i) and (ii). The proofs of statements (iii) and (iv) are similar. Denote by Y the range of the function f,
$$Y = \left\{ f(x);\ \ x \in [a,b] \right\}.$$
Hence $M = \sup Y$. From Lemma 6.13 we deduce that there exists a sequence (y_n) in Y and an increasing sequence (M_n) such that
$$M_n \leq y_n \leq M,\ \ \lim_n M_n = M.$$
The Squeezing Principle implies that
$$\lim_n y_n = M. \tag{6.8}$$
Since y_n is in the range of f there exists $x_n \in [a,b]$ such that $f(x_n) = y_n$. The sequence (x_n) is obviously bounded because it is contained in the bounded interval $[a,b]$. The Bolzano-Weierstrass Theorem (Theorem 4.29) implies that (x_n) admits a subsequence (x_{n_k}) which converges to some number x^*
$$\lim_k x_{n_k} = x^*.$$
Since $a \leq x_{n_k} \leq b$, $\forall k$, we deduce that $x^* \in [a,b]$. The continuity of f implies that
$$\lim_k y_{n_k} = \lim_k f(x_{n_k}) = f(x^*).$$
On the other hand
$$\lim_k y_{n_k} = \lim_n y_n \stackrel{(6.8)}{=} M.$$
Hence
$$M = f(x^*) < \infty.$$
□

Definition 6.15. Let $f : X \to \mathbb{R}$ be a function defined on a nonempty set $X \subset \mathbb{R}$.

(a) A point $\boldsymbol{x}_* \in X$ is called a *global minimum* of f if
$$f(x_*) \leq f(x),\ \ \forall x \in X.$$
(b) A point $\boldsymbol{x}^* \in X$ is called a *global maximum* of f if
$$f(x) \leq f(x^*),\ \ \forall x \in X.$$
□

We can rephrase Theorem 6.14 as follows.

Corollary 6.16. *A continuous function* $f : [a,b] \to \mathbb{R}$ *admits a global minimum and a global maximum.* □

Remark 6.17. The conclusions of Theorem 6.14 *do not necessarily hold* for continuous functions defined on *non-closed* intervals. Consider for example the continuous function
$$f : (0,1] \to \mathbb{R},\ \ f(x) = \frac{1}{x}.$$

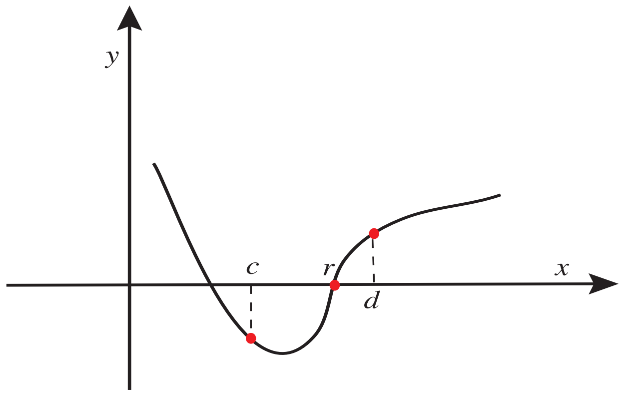

Figure 6.1. *If the graph of a continuous functions has points both below and above the x-axis, then the graph must intersect the x-axis.*

Note that $f(1/n) = n$, $\forall n \in \mathbb{N}$ so that

$$\sup\{\, f(x);\;\; x \in (0,1] \,\} = \infty. \qquad \square$$

Theorem 6.18 (The intermediate value theorem)**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a continuous function and $c, d \in [a,b]$ are real numbers such that*

$$c < d \;\; \text{and} \;\; f(c) \cdot f(d) < 0.$$

Then there exists a real number $r \in (c,d)$ such that $f(r) = 0$.

Proof. We distinguish two cases: $f(c) < 0$ of $f(c) > 0$. We discuss only the case $f(c) < 0$ depicted in Figure 6.1. The second case follows from the first case applied to the continuous function $-f$. Observe that the assumption $f(c)f(d) > 0$ implies that if $f(c) < 0$, then $f(d) > 0$.

Consider the set

$$X := \big\{ x \in [c,d];\;\; f(x) < 0 \,\big\}.$$

Clearly X is nonempty because $c \in X$. By construction, the set X is bounded above by d. Define

$$r := \sup X.$$

We will prove that $f(r) = 0$. Since $r = \sup X$, we deduce from Lemma 6.13 that there exists a sequence (x_n) in X such that $x_n \to r$ as $n \to \infty$. The function f is continuous at r so that

$$f(r) = \lim_{n\to\infty} f(x_n).$$

On the other hand $f(x_n) < 0$, for any n because $x_n \in X$. Hence $f(r) \leq 0$. In particular $r \neq d$ because $f(d) > 0$.

Figure 6.2. *The function f would be negative on $[r, r+\delta]$ if $f(r)$ were negative.*

To prove that $f(r) = 0$ it suffices to show that $f(r) \geq 0$. We argue by contradiction and we assume that $f(r) < 0$. Theorem 6.11 implies that there exists $\delta > 0$ such that if $x \in [a, b]$ and $|x - r| < \delta$, then $f(x) < 0$. Thus $f(x) < 0$ for any $x \in [a, b] \cap [r, r + \delta]$; Figure 6.2.

Choose $h > 0$ such that

$$h < \min\{ \delta, \operatorname{dist}(r, d) \}.$$

Then $r + h \in [r, d]$ and $r + h \in [r, r + \delta]$. Hence $r + h \in [c, d]$ and $f(r + h) < 0$ so that $r + h \in X$. This contradicts the fact that $r = \sup X$.

□

The Intermediate Value Theorem has many useful consequences. We present a few of them.

Corollary 6.19. *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function, $y_0 \in \mathbb{R}$ and $c \leq d$ are real numbers in the interval $[a, b]$ such that*

- *either $f(c) \leq y_0 \leq f(d)$, or*
- *$f(c) \geq y_0 \geq f(d)$.*

Then there exists $x_0 \in [c, d]$ such that $f(x_0) = y_0$.

Proof. If $f(c) = y_0$ or $f(d) = y_0$, then there is nothing to prove so we assume that $f(c), f(d) \neq y_0$. Consider the function $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - y_0$. Then $g(c)g(d) < 0$, and the Intermediate Value Theorem implies that there exists $x_0 \in (c, d)$ such that $g(x_0) = 0$, i.e., $f(x_0) = y_0$. □

Corollary 6.20. *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function and $c < d$ are real numbers in the interval $[a, b]$ such that*

$$f(x) \neq 0, \ \ \forall x \in (c, d).$$

Then the function f does not change sign in the interval (c, d), i.e., either

$$f(x) > 0, \ \ \forall x \in (c, d),$$

or

$$f(x) < 0, \ \ \forall x \in (c, d).$$

Proof. If f did change sign in the interval (c, d), then we could find two numbers $c', d' \in (c, d)$ such that $f(c') < 0$ and $f(d') > 0$. The Intermediate Value Theorem will then imply that f must equal zero at some point r situated between c' and d'. This would contradict the assumptions on f. □

Corollary 6.21. *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function,*

$$M = \sup\{ f(x); \ \ x \in [a, b] \}, \ \ m = \inf\{ f(x); \ \ x \in [a, b] \}.$$

Then the range of the function f is the interval $[m, M]$.

Proof. Observe first that

$$m \leq f(x) \leq M, \ \ \forall x \in [a,b].$$

This shows that the range of f is contained in the interval $[m,M]$. Let us now prove the opposite inclusion, i.e., $[m,M]$ is contained in the range of f. More precisely, we need to show that for any $y_0 \in [m,M]$ there exists $x_0 \in [a,b]$ such that $f(x_0) \in [m,M]$.

Observe first that Weierstrass' Theorem 6.14 implies that m, M belong to the range of f. In particular, there exist $c, d \in [a,b]$ such that $f(c) = m$ and $f(d) = M$. In particular,

$$f(c) \leq y_0 \leq f(d).$$

Corollary 6.19 implies that there exists a number x_0 situated between c and d such that $f(x_0) = y_0$. □

Corollary 6.22. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

$$\lim_{x\to\infty} f(x) \in (0,\infty] \ \ \lim_{x\to-\infty} f(x) \in [-\infty,0).$$

Then there exists $r \in \mathbb{R}$ such that $f(r) = 0$. □

The proof of this corollary is left to you as an exercise.

Corollary 6.23. *Suppose that $a < b$ and $f : [a,b] \to \mathbb{R}$ is a continuous function. Then the following statements are equivalent,*

(i) *The function f is injective.*

(ii) *The function f is strictly monotone; see Definition 5.17(v).*

Proof. The implication (ii) $\Rightarrow$ (i) is immediate. Indeed, suppose $x_1, x_2 \in [a,b]$ and $x_1 \neq x_2$. One of the numbers x_1, x_2 is smaller than the other and we can assume $x_1 < x_2$. If f is strictly increasing, then $f(x_1) < f(x_2)$, thus $f(x_1) \neq f(x_2)$. If f is strictly decreasing, then $f(x_1) > f(x_2)$ and again we conclude that $f(x_1) \neq f(x_2)$.

Let us now prove (i) $\Rightarrow$ (ii). Since $a < b$ and f is injective we deduce that either $f(a) < f(b)$, or $f(a) > f(b)$. We discuss only the first situation, $f(a) < f(b)$. The second case follows from the first case applied to the continuous injective function $g = -f$. We will prove in several steps that f is strictly increasing.

Step 1. Suppose that $d \in [a,b)$ is such that $f(d) < f(b)$. Then

$$f(d) < f(c), \ \ \forall c \in (d,b). \tag{6.9}$$

We argue by contradiction. Assume that there exists $c \in (d,b)$ such that $f(c) \leq f(d)$. Since f is injective and $d \neq c$ we deduce $f(d) \neq f(c)$ so that $f(c) < f(d)$; see Figure 6.3.

We observe that on the interval $[c,b]$ the function f has values both $< f(d)$ and $> f(d)$ because

$$f(c) < f(d) < f(b).$$

The Intermediate Value Theorem implies that there must exist a point r in the interval (c,b) such that $f(r) = f(d)$; see Figure 6.3. This contradicts the injectivity of f and completes the proof of Step 1.

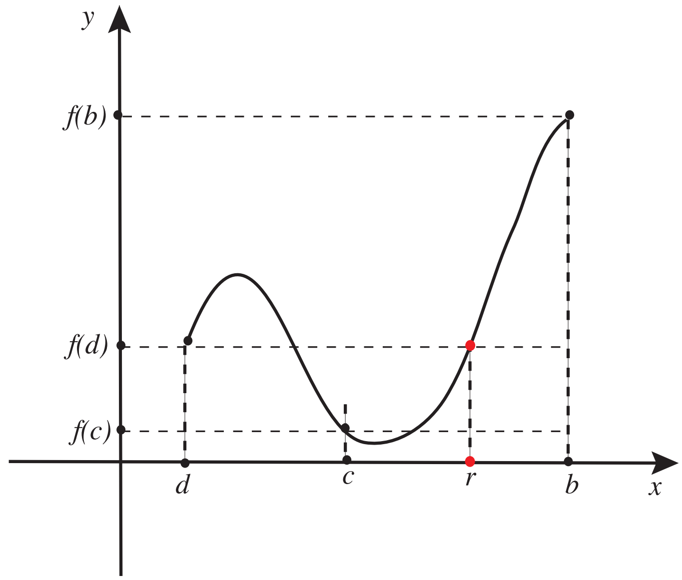

Figure 6.3. *A continuous injective function has to be monotone.*

Step 2. We will show that

$$f(c) < f(b), \quad \forall c \in (a,b). \tag{6.10}$$

Again we argue by contradiction. Assume that there exists $c \in (a,b)$ such that $f(c) \geq f(b)$. Since f is injective, $f(c) > f(b)$; Figure 6.4.

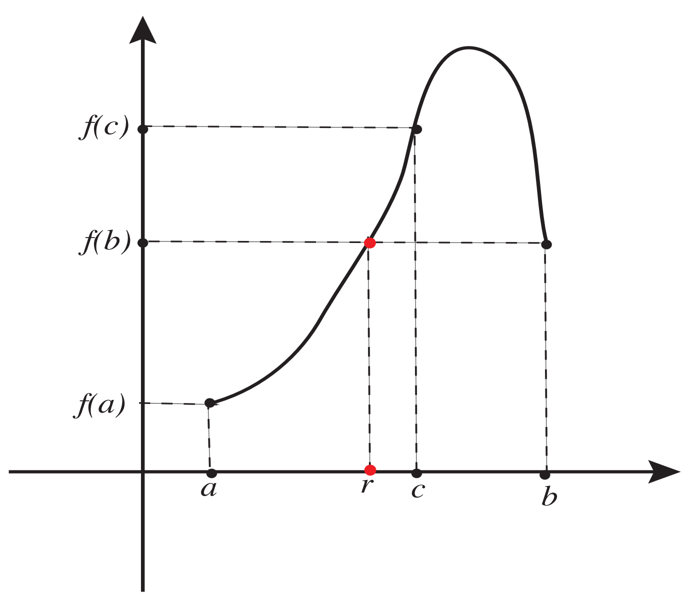

Figure 6.4. *A continuous injective function has to be monotone.*

We observe that on the interval $[a, c]$ the function f has values both $< f(b)$ and $> f(b)$ because

$$f(c) > f(b) > f(a).$$

The Intermediate Value Theorem implies that there must exist a point r in the interval (a, c) such that $f(r) = f(b)$; see Figure 6.4. This contradicts the injectivity of f and completes the proof of Step 2.

Step 3. Suppose that $d < d'$ are points in the interval (a, b). We want to show that $f(d) < f(d')$. Note that since $d \in (a, b)$ we deduce from (6.9) and (6.10) that $f(a) < f(d) < f(b)$. Since $d' \in (d, b)$ and $f(d) < f(b)$ we deduce from Step 1 that $f(d) < f(d')$. $\square$

Example 6.24. Consider the function

$$\sin : \big[-\pi/2, \pi/2\,\big] \to \mathbb{R}.$$

Using the trigonometric-circle definition of sin we deduce that the above function is strictly increasing. Note that

$$\sin(-\pi/2) = -1 = \min_{x\in\mathbb{R}} \sin x, \;\; \sin(\pi/2) = 1 = \max_{x\in\mathbb{R}} \sin x.$$

Using Corollary 6.21 we deduce that the range of this function is $[-1, 1]$ so that the resulting function

$$\sin[-\pi/2, \pi/2] \to [-1, 1]$$

is bijective. Its inverse is the function

$$\arcsin : [-1, 1] \to [-\pi/2, \pi/2].$$

We want to emphasize that, by construction, the range of arcsin is $[-\pi/2, \pi/2]$.

Similarly, the function

$$\cos : [0, \pi] \to \mathbb{R}$$

is strictly decreasing and its range is $[-1, 1]$. Its inverse is the function

$$\arccos : [-1, 1] \to [0, \pi]. \qquad \square$$

Finally, consider the function

$$\tan : (-\pi/2, \pi/2) \to \mathbb{R}.$$

Exercise 6.15 asks you to prove that the above function is bijective. Its inverse is the function

$$\arctan : \mathbb{R} \to (-\pi/2, \pi/2). \qquad \square$$

Definition 6.25. Suppose that X is a nonempty subset of the real axis and $f : X \to \mathbb{R}$ is a real valued function defined on X. The *oscillation* of the function f on the set $S \subset X$ is the quantity

$$\operatorname{osc}(f, S) := \sup_{s\in S} f(s) - \inf_{s\in S} f(s) \in [0, \infty]. \qquad \square$$

Let us observe that

$$\operatorname{osc}(f, S) = \sup_{s',s''\in S} |f(s') - f(s'')|. \tag{6.11}$$

Exercise 6.13 asks you to prove this equality.

Definition 6.26. Let $J \subset \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$ a function. We say that f is *uniformly continuous* on J if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any closed interval $I \subset J$ of length $\ell(I) \leq \delta$ we have

$$\operatorname{osc}(f, I) \leq \varepsilon. \qquad \square$$

Remark 6.27. The uniform continuity of $f : J \to \mathbb{R}$ can be alternatively characterized by the following quantized statement

$$\forall \varepsilon > 0 \;\; \exists \delta = \delta(\varepsilon) > 0 \;\text{ such that }\; \forall x, y \in J \;\; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. \qquad \square$$

Proposition 6.28. *Let $J \subset \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$ a function. If f is uniformly continuous, then f is continuous at any point $x_0 \in J$.*

Proof. Let $x_0 \in J$. We have to prove that $\forall \varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x \;\; |x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Since f is uniformly continuous, there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that, for any interval $I \subset J$ of length $\leq \delta_0(\varepsilon)$ we have $\operatorname{osc}(f, I) < \varepsilon$. Consider now the interval

$$I_{x_0} := \left\{ x \in J; \;\; |x - x_0| < \frac{\delta_0}{2} \right\}.$$

Clearly I_{x_0} has length $< \delta_0$ so that $\operatorname{osc}(f, I_{x_0}) < \varepsilon$. In particular (6.11) implies that for any $x \in I_{x_0}$ we have

$$|f(x) - f(x_0)| < \varepsilon.$$

Hence

$$|x - x_0| < \delta(\varepsilon) := \frac{\delta_0(\varepsilon)}{2} \Rightarrow x \in I_{x_0} \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

$\square$

Theorem 6.29 (Uniform Continuity)**.** *Suppose that $a < b$ are two real numbers and $f : [a, b] \to \mathbb{R}$ is a continuous function. Then f is uniformly continuous, i.e., for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any interval $I \subset [a, b]$ of length $\ell(I) \leq \delta$ we have*

$$\operatorname{osc}(f, I) \leq \varepsilon.$$

Proof. We have to prove that

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall I \subset [a, b] \;\text{ interval}, \;\; \ell(I) \leq \delta \Rightarrow \operatorname{osc}(f, I) \leq \varepsilon.$$

We argue by contradiction and we assume that the opposite is true

$$\exists \varepsilon_0 > 0 \;\; \forall \delta > 0 \;\; \exists I = I_\delta \subset [a, b] \;\text{ interval}, \;\; \ell(I_\delta) \leq \delta \wedge \operatorname{osc}(f, I_\delta) > \varepsilon_0.$$

We deduce that ***for any*** $n \in \mathbb{N}$ there exists a closed interval $I_n = [a_n, b_n] \subset [a, b]$ of length $\leq \frac{1}{n}$ such that

$$\operatorname{osc}(f, [a_n, b_n]) > \varepsilon_0. \tag{6.12}$$

Since the length of $[a_n, b_n]$ is $\leq \frac{1}{n}$ we deduce

$$a_n < b_n \leq a_n + \frac{1}{n}.$$

The Bolzano-Weierstrass Theorem 4.29 implies that the sequence (a_n) admits a convergent subsequence (a_{n_k}). We set

$$a_* := \lim_{k\to\infty} a_{n_k}.$$

Since $a \leq a_n \leq b$, we deduce $a_* \in [a,b]$. Since

$$a_{n_k} < b_{n_k} \leq a_{n_k} + \frac{1}{n_k}$$

we deduce from the Squeezing Principle that

$$\lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty} a_{n_k} = a_*.$$

On the other hand, since $a_* \in [a,b]$, the function f is continuous at a_*. Thus there exists $\delta > 0$ such that

$$|x - a_*| < \delta \Rightarrow |f(x) - f(a_*)| < \frac{\varepsilon_0}{4}.$$

In other words,

$$\operatorname{dist}(x, a_*) < \delta \Rightarrow f(a_*) - \frac{\varepsilon_0}{4} < f(x) < f(a_*) + \frac{\varepsilon_0}{4}.$$

Since $a_{n_k}, b_{n_k} \to a_*$ there exists k_0 such that

$$[a_{n_{k_0}}, b_{n_{k_0}}] \subset (a_* - \delta, a_* + \delta) \Rightarrow f(a_*) - \frac{\varepsilon_0}{4} < f(x) < f(a_*) + \frac{\varepsilon_0}{4}, \;\; \forall x \in [a_{n_{k_0}}, b_{n_{k_0}}].$$

Thus

$$f(a_*) - \frac{\varepsilon_0}{4} \leq \inf_{x\in[a_{n_{k_0}}, b_{n_{k_0}}]} f(x) \leq \sup_{x\in[a_{n_{k_0}}, b_{n_{k_0}}]} f(x) \leq f(a_*) + \frac{\varepsilon_0}{4}.$$

This shows that

$$\operatorname{osc}\big(\, f,\; [a_{n_{k_0}}, b_{n_{k_0}}]\,\big) \leq \left(f(a_*) + \frac{\varepsilon_0}{4} \right) - \left(f(a_*) - \frac{\varepsilon_0}{4} \right) = \frac{\varepsilon_0}{2}.$$

This contradicts (6.12) and completes the proof of the theorem. □

Remark 6.30. The above result is no longer valid for continuous functions defined on *non-closed* or *unbounded* intervals. Consider for example the continuous function $f : (0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. For each $n \in \mathbb{N}$, $n > 1$ we define

$$I_n = \Big[\frac{1}{n+1}, \frac{1}{n}\Big].$$

Since f is decreasing we deduce that

$$\sup_{x\in I_n} f(x) = f\Big(\frac{1}{n+1}\Big) = n+1, \;\; \inf_{x\in I_n} f(x) = f\Big(\frac{1}{n}\Big) = n$$

so that $\operatorname{osc}(f, I_n) = 1$. On the other hand, $\ell(I_n) = \frac{1}{n(n+1)} \to 0$ as $n \to \infty$. We have thus produced arbitrarily short intervals over which the oscillation is 1.

Exercise 6.12 describes an example of continuous function over an *unbounded* interval that is *not* uniformly continuous on that interval. □

6.3. Exercises

Exercise 6.1. Prove Theorem 6.2. □

Exercise 6.2. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are two continuous functions such that $f(q) = g(q)$, $\forall q \in \mathbb{Q}$. Prove that $f(x) = g(x)$, $\forall x \in \mathbb{R}$.
Hint. You may want to invoke Proposition 3.32. □

Exercise 6.3. Prove Corollary 6.3. □

Exercise 6.4. Prove the inequality (6.1). □

Exercise 6.5. Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous functions.
(a) Prove that the function $|f|$ continuous.
(b) Prove that for any $x \in [a, b]$ we have

$$\max\{\, f(x), g(x)\,\} = \frac{1}{2}\big(f(x) + g(x) + |f(x) - g(x)|\,\big).$$

(c) Prove that the function $h : [a, b] \to \mathbb{R}$, $h(x) = \max\{f(x), g(x)\}$ is continuous. □

Exercise 6.6 (Weierstrass)**.** Suppose that X is a nonempty set of real numbers, $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of functions, and $f : X \to \mathbb{R}$ a function on X. Suppose that for any $n \in \mathbb{N}$ we have

$$M_n := \sup_{x \in X} |f_n(x) - f(x)| < \infty.$$

Prove that the following statements are equivalent.

(i) The sequence (f_n) converges uniformly to f on X.
(ii) $\lim_{n\to\infty} M_n = 0$.

□

Exercise 6.7 (Weierstrass)**.** Consider a sequence of functions $f_n : [a, b] \to \mathbb{R}$, $n \geq 0$, where a, b are real numbers $a < b$. Suppose that there exists a sequence of positive real numbers $(c_n)_{n \geq 0}$ with the following properties.

(i) $|f_n(x)| \leq c_n$, $\forall n \geq 0$, $\forall x \in [a, b]$.
(ii) The series $\sum_{n \geq 0} c_n$ is convergent.

(a) Prove that for any $x \in [a, b]$, the series of real numbers $\sum_{n \geq 0} f_n(x)$ is absolutely convergent. Denote by $s(x)$ its sum.
(b) Denote by $s_n(x)$ the n-th partial sum

$$s_n(x) = f_0(x) + f_1(x) + \cdots + f_n(x)$$

Prove that the sequence of functions $s_n : [a, b] \to \mathbb{R}$ converges uniformly on $[a, b]$ to the function $s : [a, b] \to \mathbb{R}$ defined in (a).

Hint. Use Exercise 6.6. □

Exercise 6.8. Consider the power series

$$\sum_{n\geq 0} a_n x^n, \quad a_n \in \mathbb{R}. \tag{6.13}$$

Suppose that for some $R > 0$ the series

$$\sum_{n\geq 0} a_n R^n$$

is absolutely convergent.

(a) Prove that the series (6.13) converges absolutely for any $x \in [-R, R]$. Denote by $s(x)$ its sum.

(b) Denote by $s_n(x)$ the n-th partial sum

$$s_n(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Prove that the resulting sequence of functions $s_n : [-R, R] \to \mathbb{R}$ converges uniformly to $s(x)$. Conclude that the function $s(x)$ is continuous on $[-R, R]$.

Hint. Use the results in Exercise 6.7. □

Exercise 6.9. Consider the sequence of functions

$$f_n : [0, 1] \to \mathbb{R}, \quad f_n(x) = x^n, \quad n \in \mathbb{N}.$$

(a) Prove that for any $x \in [0, 1]$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ is convergent. Compute its limit $f(x)$.

(b) Given $n \in \mathbb{N}$ compute

$$\sup_{x\in[0,1]} |f_n(x) - f(x)|.$$

(c) Prove that the sequence of functions $f_n(x)$ does *not* converge uniformly to the function $f(x)$ defined in (a). □

Exercise 6.10. (a) Prove Corollary 6.22.

(b) Let $f(x)$ be a polynomial of *odd* degree. Prove that there exists $r \in \mathbb{R}$ such that $f(r) = 0$.□

Exercise 6.11. Suppose that $f : [0, 1] \to [0, 1]$ is a continuous function. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$. Can you give a geometric interpretation of this result? □

Exercise 6.12. (a) Find the oscillation of the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^2$, over an interval $[a, b] \subset (0, \infty)$.

(b) Prove that for any $n \in \mathbb{N}$ one can find an interval $[a, b] \subset [0, \infty)$ of length $\leq \frac{1}{n}$ over which the oscillation of f is ≥ 1. □

Exercise 6.13. (a) Suppose that $f : X \to \mathbb{R}$ is a function defined on a set X, and $Y \subset X$. Prove that

$$\operatorname{osc}(f, X) = \sup_{x',x''\in X} |f(x') - f(x'')| \ \text{ and } \ \operatorname{osc}(f, Y) \leq \operatorname{osc}(f, X).$$

(b) Consider a function $f:(a,b)\to\mathbb{R}$. Prove that f is continuous at a point $x_0\in(a,b)$ if and only if

$$\lim_{\delta\searrow 0}\operatorname{osc}\big(f,[x_0-\delta,x_0+\delta]\big)=0.$$

(c) Suppose that $f:[a,b]\to\mathbb{R}$ is a continuous function. Prove that

$$\operatorname{osc}(f,\ (a,b)\,)=\operatorname{osc}(f,\ [a,b]\,).$$

Exercise 6.14 (Cauchy)**.** Suppose $X\subset\mathbb{R}$ is a nonempty set of real number and $f_n:X\to\mathbb{R}$ is a sequence of real valued functions defined on X. Prove that the following statements are equivalent.

(i) There exists a function $f:X\to\mathbb{R}$ such that the sequence $f_n:X\to\mathbb{R}$ converges uniformly on X to $f:X\to\mathbb{R}$.

(ii) $\forall\varepsilon>0$, $\exists N=N(\varepsilon)\in\mathbb{N}$ such that

$$\forall n,m>N(\varepsilon),\ \ \forall x\in X:\ \ |f_n(x)-f_m(x)|<\varepsilon.$$

□

Exercise 6.15. Consider the function

$$f:(-\pi/2,\pi/2)\to\mathbb{R},\ \ f(x)=\tan x=\frac{\sin x}{\cos x}.$$

Prove that f is strictly increasing and

$$\lim_{x\to\pm\pi/2}f(x)=\pm\infty.$$

Conclude that f is bijective. □

6.4. Exercises for extra-credit

Exercise* 6.1. Suppose that $f:[a,b]\to\mathbb{R}$ is a continuous function. For any $x\in[a,b]$ we define

$$m(x)=\inf_{t\in[a,x]}f(x),\ \ M(x)=\sup_{t\in[a,x]}f(x).$$

Prove that the functions $x\mapsto m(x)$ and $x\mapsto M(x)$ are continuous. □

Exercise* 6.2. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying

$$f(0)=0,\ \ f(1)=1$$

and

$$f(x+y)=f(x)+f(y),\ \ \forall x\in\mathbb{R}.$$

Prove that $f(x)=x$, $\forall x\in\mathbb{R}$. □

Exercise* 6.3. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a *continuous* function satisfying the following properties.

(i) $f(x)>0$, $\forall x\in\mathbb{R}$.

(ii) $f(x+y)=f(x)f(y)$, $\forall x,y\in\mathbb{R}$.

Set $a := f(1)$. Prove that $f(x) = a^x$, $\forall x \in \mathbb{R}$. $\square$

Exercise* 6.4 (Dini)**.** Suppose that $f_n : [0,1] \to \mathbb{R}$, $n \in \mathbb{N}$ is a sequence of continuous functions with the following properties.

(i) For any $t \in [0,1]$ we have

$$f_1(t) \leq f_2(t) \leq f_3(t) \leq \cdots .$$

(ii) There exists a continuous function $f : [0,1] \to \mathbb{R}$ such that

$$\lim_{n\to\infty} f_n(t) = f(t).$$

Prove that the sequence of functions (f_n) converges *uniformly* to f on $[0,1]$. $\square$

Exercise* 6.5. Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous function. For $n \in \mathbb{R}$ define

$$f_n : [0,1] \to \mathbb{R}$$

by setting

$$f_n(x) := \begin{cases} f(0), & \text{if } x = 0 \\ \min\bigl\{ f(x); \frac{k-1}{n} \leq x \leq \frac{k}{n} \bigr\}, & \text{if } \frac{k-1}{n} < x \leq \frac{k}{n}, \ \ k = 1, \dots, n. \end{cases}$$

Prove that the sequence of functions (f_n) converges *uniformly* to the function f on $[0,1]$. $\square$

Chapter 7

Differential calculus

7.1. Linear approximation and derivative

The differential calculus is one of the most consequential scientific discoveries in the history of mankind. Surprisingly, this revolutionary theory is based on a very simple principle: often one can learn nontrivial things about complicated objects by approximating them with simpler ones.

In the case at hand, the complicated object is a function $f : (a, b) \to \mathbb{R}$ and one would like to understand its behavior near a point $x_0 \in (a, b)$. To achieve this, we try to approximate f with a simpler function, and the linear functions are the simplest nontrivial candidates.

Definition 7.1. Suppose that I is an interval[1] on the real axis, $f : I \to \mathbb{R}$ is a function and $x_0 \in I$. A *linear approximation* or *linearization* of f at x_0 is a linear function

$$L : \mathbb{R} \to \mathbb{R}, \ \ L(x) = b + m(x - x_0)$$

such that

$$L(x_0) = f(x_0) \tag{7.1}$$

and

$$f(x) - L(x) = o(x - x_0) \ \text{ as } x \to x_0. \tag{7.2}$$

Above we used Landau's symbol o defined in (5.35) signifying that

$$\lim_{x \in I,\, x \to x_0} \frac{f(x) - L(x)}{x - x_0} = 0.$$

The function is said to be *linearizable* at x_0 if it admits a linearization at x_0. □

Suppose that L is a linearization of the function $f : I \to \mathbb{R}$ at x_0. By (7.1), the value of L at x_0 is equal to the value of f at x_0, $L(x_0) = f(x_0)$. On the other hand

$$L(x_0) = b + m(x_0 - x_0) = b$$

and we deduce that $L(x)$ has the form

$$L(x) = f(x_0) + m(x - x_0).$$

[1]The interval I could be closed, could be open, could be neither, could be bounded or not.

The linear function $L(x)$ is meant to approximate the function $f(x)$ for x not too far for x_0. The *error* of this linear approximation of $f(x)$ is the difference $r(x) = f(x) - L(x)$ which by definition is $o(x - x_0)$ as $x \to x_0$. In less rigorous terms, $r(x)$ is a tiny fraction of $(x - x_0)$ when x is close to x_0. Note that

$$f(x) - L(x) = f(x) - \big(m(x - x_0) + f(x_0) \big) = f(x) - f(x_0) - m(x - x_0),$$

$$\frac{f(x) - L(x)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - m.$$

Since

$$0 = \lim_{x \to x_0} \frac{f(x) - L(x)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - m$$

we deduce that

$$m = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{7.3}$$

Thus if f is linearizable at x_0, then there exists a *unique* linearization $L(x)$ described by

$$L(x) = f(x_0) + m(x - x_0),$$

where the slope m is given by (7.3).

Definition 7.2. Suppose that I is an interval of the real axis, $f : I \to \mathbb{R}$ is a function and $x_0 \in I$.

(i) We say that f is *differentiable at* x_0 if the limit (7.3)

$$\lim_{\substack{x \to x_0 \\ x \in I}} \frac{f(x) - f(x_0)}{x - x_0} \tag{7.4}$$

exists and it is finite. If this is the case, we denote the limit by $f'(x_0)$ or $\frac{df}{dx}|_{x=x_0}$ and we will refer to it as the *derivative* of f at x_0.

(ii) We say that f is differentiable on I if it is differentiable at *any* point $x \in I$. The function $f' : I \to \mathbb{R}$ that assigns to $x \in I$ the derivative $f'(x)$ of f at x is called the *derivative of the function f on the interval I.* □

Remark 7.3. In concrete computations it is often convenient to describe the derivative of f at x_0 as the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

This is obtained from (7.4) if we denote by h the "displacement" $x - x_0$. With this notation we have $x = x_0 + h$ and

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{f(x_0 + h) - f(x_0)}{h}. \qquad \square$$

The next result summarizes the observations we have made so far.

Proposition 7.4. *Suppose that I is an interval of the real axis, $f : I \to \mathbb{R}$ is a function and $x_0 \in I$. Then the following statements are equivalent.*

(i) *The function f is differentiable at x_0.*

(ii) *The function f is linearizable at x_0.*

(iii) *The function f is differentiable at x_0 and the function $L(x) = f(x_0)+f'(x_0)(x-x_0)$ is the linearization of f at x_0, i.e.,*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + r(x), \quad r(x) = o(x - x_0) \;\; \text{as } x \to x_0. \tag{7.5}$$

□

We should perhaps give a geometric interpretation to the linear approximation of f at x_0. The graph of f is the curve

$$G_f := \Big\{ \big(x, f(x)\big) \in \mathbb{R}^2;\;\; x \in (a, b) \Big\}.$$

The point $x_0 \in I$ determines a point $P_0 = (x_0, f(x_0))$ on the curve G_f; see Figure 7.1.

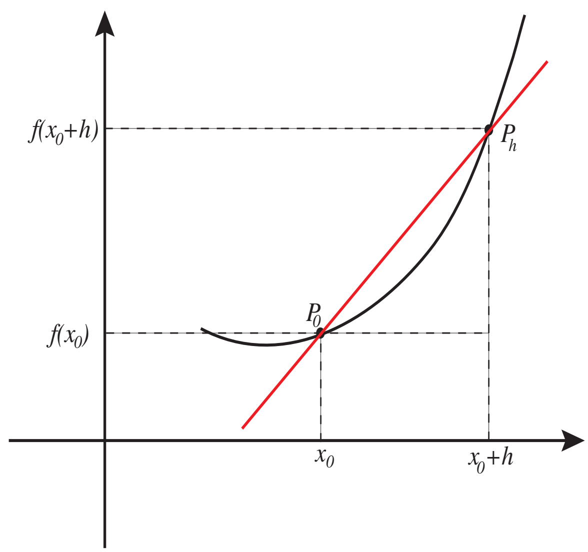

Figure 7.1. *A tangent line to the graph of a function is a limit of secant lines.*

The graph of a linear function $L(x)$ is a line in the plane and since we are interested in approximating the behavior of f near x_0 it makes sense to look only at lines $\ell_{P_0,P}$ determined by two points P_0, P on the graph G_f. Since we are interested only in the behavior of f *near* x_0, we may assume that the the point P is not too far from P_0. Thus we assume that the coordinates of P are $(\, x_0 + h, f(x_0 + h)\,)$, where h is very small.

In more concrete terms, we look at the lines ℓ_{P_0,P_h} determined by the two points

$$P_0 := (\, x_0, f(x_0)\,), \;\; P_h := (\, x_0 + h, f(x_0 + h)\,),$$

where h very small. The slope of the line ℓ_{P_0,P_h} is

$$m(h) := \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} = \frac{f(x_0 + h) - f(x_0)}{h},$$

so its equation is

$$y - f(x_0) = m(h)(x - x_0).$$

This is the graph of the linear function

$$L_{x_0,h}(x) = f(x_0) + m(h)(x - x_0).$$

Suppose that as $h \to 0$ the line ℓ_{P_0,P_h} stabilizes to some limiting position. This limit line goes through the point P_0 and therefore its position is determined by its slope

$$\lim_{h\to 0} m(h) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

We see that this limit exists and it is finite if and only if f is differentiable at x_0. In this case, the limit line is the graph of the linear approximation of f at x_0.

Definition 7.5. Suppose $I \subset \mathbb{R}$ is an interval of the real axis and $f : I \to \mathbb{R}$ is a function differentiable at x_0. The *tangent line to the graph* of f at x_0 is the graph of the linearization of f at x_0. □

Remark 7.6. (a) The quantities

$$\frac{f(x) - f(x_0)}{x - x_0}, \quad \frac{f(x_0+h) - f(x_0)}{h}$$

are called *difference quotients* of f at x_0. You should think of such a difference quotient as measuring the average rate of change of the quantity f over the interval $[x_0, x]$.

In physics, the numerator $f(x) - f(x_0)$ is denoted by Δf while the denominator is denoted Δx. The symbol Δ is shorthand for "*variation in*". Thus

$$\frac{df}{dx} = \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}.$$

From the equality

$$f'(x) = \frac{df}{dx}$$

we deduce formally

$$df = f'(x)dx.$$

The expression $f'(x)dx$ is called the *differential* of f and as the above equality suggests, it is denoted by df.

(b) Often a function $f : [a, b] \to \mathbb{R}$ has a physical meaning. For example, the interval $[a, b]$ can signify a stretch of highway between mile a and mile b and $f(x)$ could be the temperature at mile x and thus it is measured in $^\circ F$. The difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0}$$

has a different meaning. The numerator $f(x) - f(x_0)$ describes the change in temperature from mile x_0 to mile x and it is again measured in $^\circ F$, while the numerator $x - x_0$ is the "distance" (could be negative) from mile x_0 to mile x and thus it is measured in miles. We deduce that the quotient is measured in different units, degrees-per-mile, and should be viewed as the average rate of change in temperature per mile. When $x \to x_0$ we are measuring the rate of change in temperature over shorter and shorter stretches of highway. For this reason, the limit $f'(x_0)$ is sometimes referred to as the *infinitesimal rate of change*. □

The differentiability of a function at a point x_0 imposes restrictions on the behavior of the function near that point. Our next elementary result describes one such restriction. Its proof is left to you as an exercise.

Proposition 7.7. *Suppose I is an interval of the real axis $\mathbb{R}$ and $f : I \to \mathbb{R}$ is a function that is differentiable at a point $x_0 \in I$. Then f is continuous at x_0, i.e.,*

$$\lim_{I \ni x \to x_0} f(x) = f(x_0). \qquad \square$$

Remark 7.8. The converse of the above result is not true. There exist continuous functions $f : [0,1] \to \mathbb{R}$ which are nowhere differentiable. For example, the function

$$f : [0,1] \to \mathbb{R}, \quad f(t) = \sum_{n=0}^{\infty} \frac{\cos(5^n t)}{2^n},$$

is continuous and nowhere differentiable. Its graph, depicted in Figure 7.2, may convince you of the validity of this claim. The rigorous proof of this fact is rather ingenious and for details and generalizations we refer to [6]. $\square$

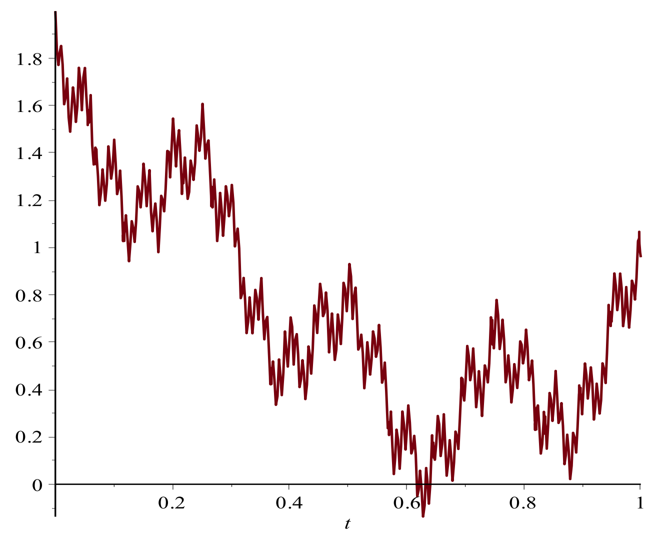

Figure 7.2. *Weierstrass's example of continuous, nowhere differentiable function.*

Suppose that $I \subset \mathbb{R}$ is an interval and $f : I \to \mathbb{R}$ is a differentiable function. We say that f is *twice* differentiable if its derivative f', viewed as a function $f' : I \to \mathbb{R}$, is also differentiable. The *second derivative* of f denoted by f'' or $\frac{d^2 f}{dx^2}$ is the derivative of f'

$$f'' := \frac{d}{dx}(f').$$

Recursively, for any natural number $n > 1$, we say that f is *n-times differentiable* if its derivative is $(n-1)$-times differentiable. The n-th derivative of f is the function $f^{(n)} : I \to \mathbb{R}$ defined recursively as

$$f^{(n)} := \frac{d}{dx}\big(f^{(n-1)}\big).$$

Often we will use the alternate notation $\frac{d^n f}{dx^n}$ to denote the n-th derivative of f.

Definition 7.9. Let $I \subset \mathbb{R}$ be an interval.

(i) We denote by $C^0(I)$ the set consisting of all the continuous functions $f : I \to \mathbb{R}$.

(ii) If n is a natural number, then we denote by $C^n(I)$ the space of functions $f : I \to \mathbb{R}$ which are
- n-times differentiable and
- the n-th derivative $f^{(n)}$ is a continuous function.

We will refer to the functions in $C^n(I)$ as C^n*-functions*.

(iii) We denote by $C^\infty(I)$ the space of functions $I \to \mathbb{R}$ which are infinitely many times differentiable. We will refer to such functions as *smooth*.

□

7.2. Fundamental examples

In this section we describe a *very important* collection of differentiable functions.

Example 7.10 (Constant functions)**.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is the function which is identically equal to a fixed real number c,

$$f(x) = c, \quad \forall x \in \mathbb{R}.$$

Then f is differentiable and $f'(x) = 0$, $\forall x \in \mathbb{R}$. Indeed, for any $x_0 \in \mathbb{R}$

$$\frac{f(x_0 + h) - f(x_0)}{h} = 0, \quad \forall h \neq 0.$$

□

Example 7.11 (Monomials)**.** Suppose that $n \in \mathbb{N}$ and consider the *monomial function* $\mu_n : \mathbb{R} \to \mathbb{R}$, $\mu_n(x) = x^n$. Then μ_n is differentiable on $\mathbb{R}$ and its derivative is

$$\mu_n'(x) = nx^{n-1}, \quad \forall x \in \mathbb{R} \Longleftrightarrow \frac{d}{dx}(x^n) = nx^{n-1}. \tag{7.6}$$

To prove this claim we investigate the difference quotients of μ_n at $x_0 \in \mathbb{R}$. We have

$$\mu_n(x_0 + h) - \mu_n(x_0) = (x_0 + h)^n - x_0^n$$

(use Newton's binomial formula (3.6))

$$= x_0^n + \binom{n}{1} x_0^{n-1} h + \binom{n}{2} x_0^{n-2} h^2 + \cdots + \binom{n}{n} h^n - x_0^n$$

$$= h \left(\binom{n}{1} x_0^{n-1} + \binom{n}{2} x_0^{n-2} h + \cdots + \binom{n}{n} h^{n-1} \right),$$

so that

$$\frac{\mu_n(x_0 + h) - \mu_n(x_0)}{h} = \binom{n}{1} x_0^{n-1} + \binom{n}{2} x_0^{n-2} h + \cdots + \binom{n}{n} h^{n-1}.$$

Now observe that

$$\mu_n'(x_0) = \lim_{h \to 0} \frac{\mu_n(x_0 + h) - \mu_n(x_0)}{h}$$

$$= \lim_{h \to 0} \left(\binom{n}{1} x_0^{n-1} + \binom{n}{2} x_0^{n-2} h + \cdots + \binom{n}{n} h^{n-1} \right) = \binom{n}{1} x_0^{n-1} = nx_0^{n-1}.$$

For example

$$(x^2)' = 2x, \;\; d(x^2) = 2xdx. \qquad \square$$

Example 7.12 (Power functions)**.** Fix a real number α and consider the power function

$$f:(0,\infty)\to\mathbb{R}, \;\; f(x) = x^\alpha.$$

Then f is differentiable and its derivative is

$$f'(x) = \alpha x^{\alpha-1}, \;\; \forall x>0 \Longleftrightarrow \frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}. \tag{7.7}$$

To prove this claim we investigate the difference quotients of $f(x)$ at $x_0\in(0,\infty)$. We have

$$(x_0+h)^\alpha - x_0^\alpha = \left(x_0\Big(1+\frac{h}{x_0}\Big)\right)^\alpha - x_0^\alpha = x_0^\alpha\left(\Big(1+\frac{h}{x_0}\Big)^\alpha - 1\right),$$

$$\frac{f(x_0+h)-f(x_0)}{h} = x_0^\alpha\frac{\Big(1+\frac{h}{x_0}\Big)^\alpha - 1}{h} = x_0^\alpha\frac{\Big(1+\frac{h}{x_0}\Big)^\alpha - 1}{x_0\frac{h}{x_0}}$$

$$= x_0^{\alpha-1}\frac{\Big(1+\frac{h}{x_0}\Big)^\alpha - 1}{\frac{h}{x_0}}.$$

We set $t:=\frac{h}{x_0}$ and we observe that $t\to 0$ as $h\to 0$ and

$$\frac{f(x_0+h)-f(x_0)}{h} = x_0^{\alpha-1}\frac{(1+t)^\alpha-1}{t}.$$

Invoking the fundamental limit (5.21) we deduce

$$\lim_{t\to 0}\frac{(1+t)^\alpha-1}{t} = \alpha$$

so that

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \alpha x_0^{\alpha-1}.$$

Note that if $\alpha=\frac{1}{2}$, then $f(x)=\sqrt{x}$ and we deduce

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}, \;\; d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}. \tag{7.8}$$

$\square$

Example 7.13 (The exponential function)**.** Consider the exponential function

$$f:\mathbb{R}\to\mathbb{R}, \;\; f(x) = e^x.$$

This function is differentiable and its derivative is

$$f'(x) = e^x, \;\; \forall x\in\mathbb{R} \Longleftrightarrow \frac{d}{dx}(e^x) = e^x. \tag{7.9}$$

To prove this claim we investigate the difference quotients of f at $x_0\in\mathbb{R}$. We have

$$f(x_0+h)-f(x_0) = e^{x_0+h} - e^{x_0} = e^{x_0}(e^h-1),$$

$$\frac{f(x_0+h)-f(x_0)}{h} = e^{x_0}\frac{e^h-1}{h}.$$

On the other hand, the fundamental limit (5.20) implies that

$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$

Hence

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=e^{x_0}.$$

These computations show that the exponential function is smooth, i.e., infinitely many times differentiable and

$$\frac{d^n}{dx^n}e^x=e^x,\quad d(e^x)=e^x dx. \tag{7.10}$$

□

Example 7.14 (The natural logarithm)**.** Consider the natural logarithm

$$f:(0,\infty)\to\mathbb{R},\quad f(x)=\ln x=\log x.$$

Then f is differentiable and its derivative is

$$f'(x)=\frac{1}{x},\quad \forall x>0 \Longleftrightarrow \frac{d}{dx}(\ln x)=\frac{1}{x},\quad d(\ln x)=\frac{dx}{x}. \tag{7.11}$$

To prove this claim we investigate the difference quotients of f at $x_0>0$. We have

$$f(x_0+h)-f(x_0)=\ln(x_0+h)-\ln x_0=\ln\left(x_0\Big(1+\frac{h}{x_0}\Big)\right)-\ln x_0$$

$$=\ln x_0+\ln\Big(1+\frac{h}{x_0}\Big)-\ln x_0=\ln\Big(1+\frac{h}{x_0}\Big),$$

$$\frac{f(x_0+h)-f(x_0)}{h}=\frac{\ln\Big(1+\frac{h}{x_0}\Big)}{h}=\frac{\ln\Big(1+\frac{h}{x_0}\Big)}{x_0\frac{h}{x_0}}=\frac{1}{x_0}\,\frac{\ln\Big(1+\frac{h}{x_0}\Big)}{\frac{h}{x_0}}.$$

We set $t=\frac{h}{x_0}$ and we conclude from above that

$$\frac{f(x_0+h)-f(x_0)}{h}=\frac{1}{x_0}\,\frac{\ln(1+t)}{t}.$$

Note that t goes to zero when $h\to 0$. We can now invoke (5.19) to conclude that

$$\lim_{t\to 0}\frac{\ln(1+t)}{t}=1.$$

This proves

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\frac{1}{x_0}.$$

□

Example 7.15 (Trigonometric functions)**.** The trigonometric functions

$$\sin,\cos:\mathbb{R}\to\mathbb{R}$$

are differentiable and

$$\frac{d}{dx}(\sin x)=\cos x,\quad \frac{d}{dx}(\cos x)=-\sin x. \tag{7.12}$$

Fix $x_0\in\mathbb{R}$. We have

$$\sin(x_0+h)-\sin x_0\stackrel{(5.33a)}{=}\sin x_0\cos h+\cos x_0\sin h-\sin x_0$$

$$=\sin x_0\big(\cos h-1\big)+\cos x_0\sin h=-2\sin^2(h/2)\sin x_0+\cos x_0\sin h.$$

Hence

$$\frac{\sin(x_0+h)-\sin x_0}{h} = -2\sin x_0 \frac{\sin^2(h/2)}{h} + \cos x_0 \frac{\sin h}{h}.$$

$$= -\sin x_0 \frac{\sin^2(h/2)}{\frac{h}{2}} + \cos x_0 \frac{\sin h}{h} = -\frac{h}{2}\sin x_0 \left(\frac{\sin(\frac{h}{2})}{\frac{h}{2}}\right)^2 + \cos x_0 \frac{\sin h}{h}.$$

From the fundamental identity (5.27) we deduce that

$$\lim_{t\to 0}\frac{\sin t}{t} = 1.$$

Hence

$$\lim_{h\to 0}\frac{h}{2}\sin x_0 \left(\frac{\sin(\frac{h}{2})}{\frac{h}{2}}\right)^2 = 0, \quad \lim_{h\to 0}\cos x_0 \frac{\sin h}{h} = \cos x_0,$$

and thus

$$\lim_{h\to 0}\frac{\sin(x_0+h)-\sin x_0}{h} = \cos x_0.$$

The equality

$$\lim_{h\to 0}\frac{\cos(x_0+h)-\cos x_0}{h} = -\sin x_0$$

is proved in a similar fashion and the details are left to you as an exercise. □

7.3. The basic rules of differential calculus

In the previous section we have computed the derivatives of a few important functions. In this section we describe a few basic rules which will allow us to easily compute the derivatives of almost any function.

Theorem 7.16 (Arithmetic rules of differentiation)**.** *Suppose that $I \subset \mathbb{R}$ is an interval, and $f, g : I \to \mathbb{R}$ are two functions differentiable at x_0. Then the following hold.*

Addition. *The sum $f+g$ is differentiable at x_0 and*

$$\boxed{(f+g)'(x_0) = f'(x_0) + g'(x_0).}$$

Scalar multiplication. *If c is a real number, then the function cf is differentiable at x_0 and*

$$\boxed{(cf)'(x_0) = cf'(x_0).}$$

Product. *The product $f \cdot g$ is differentiable at x_0 and its derivative is given by the* product rule *or* Leibniz rule

$$\boxed{(f\cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).}$$

Quotient. *If $g(x_0) \neq 0$, then there exists $\delta > 0$ such that*

$$\forall x \in I \ \ |x - x_0| < \delta \Rightarrow g(x) \neq 0.$$

Set

$$I_{x_0,\delta} := \big\{ x \in I; \ \ |x - x_0| < \delta \big\}.$$

The quotient $\frac{f}{g}$ is a well defined function on $I_{x_0,\delta}$ which is differentiable at x_0 and its derivative at x_0 is determined by the quotient rule

$$\boxed{\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.}$$

Proof. Addition. We have

$$\begin{aligned}\frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} &= \frac{f(x_0+h) - f(x_0) + g(x_0+h) - g(x_0)}{h}\\ &= \frac{f(x_0+h) - f(x_0)}{h} + \frac{g(x_0+h) - g(x_0)}{h}.\end{aligned}$$

Hence

$$\begin{aligned}\lim_{h\to 0}\frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} &= \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h} + \lim_{h\to 0}\frac{g(x_0+h) - g(x_0)}{h}\\ &= f'(x_0) + g'(x_0).\end{aligned}$$

Scalar multiplication. We have

$$\frac{(cf)(x_0+h) - (cf)(x_0)}{h} = c\frac{f(x_0+h) - f(x_0)}{h}$$

so that

$$\lim_{h\to 0}\frac{(cf)(x_0+h) - (cf)(x_0)}{h} = c\lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h} = cf'(x_0).$$

Product. We have

$$\begin{aligned}(f\cdot g)(x_0+h) - (f\cdot g)(x_0) &= f(x_0+h)g(x_0+h) - f(x_0)g(x_0)\\ &= f(x_0+h)g(x_0+h) - f(x_0)g(x_0+h) + f(x_0)g(x_0+h) - f(x_0)g(x_0)\\ &= \big(f(x_0+h) - f(x_0)\big)g(x_0+h) + f(x_0)\big(g(x_0+h) - g(x_0)\big),\end{aligned}$$

so that

$$\frac{(f\cdot g)(x_0+h) - (f\cdot g)(x_0)}{h} = \frac{\big(f(x_0+h) - f(x_0)\big)}{h}g(x_0+h) + f(x_0)\frac{\big(g(x_0+h) - g(x_0)\big)}{h}.$$

Since g is differentiable at x_0 it is also continuous at x_0 by Proposition 7.7. Hence

$$\lim_{h\to 0} g(x_0+h) = g(x_0),\quad \lim_{h\to 0} f(x_0)\frac{\big(g(x_0+h) - g(x_0)\big)}{h} = f(x_0)g'(x_0).$$

Since f is differentiable at x_0 we deduce

$$\lim_{h\to 0}\frac{\big(f(x_0+h) - f(x_0)\big)}{h}g(x_0+h) = \lim_{h\to 0}\frac{\big(f(x_0+h) - f(x_0)\big)}{h}\cdot\lim_{h\to 0} g(x_0+h) = f'(x_0)g(x_0).$$

Hence

$$\lim_{h\to 0}\frac{(f\cdot g)(x_0+h) - (f\cdot g)(x_0)}{h} = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Quotient. The function g is differentiable at x_0, thus continuous at this point. From Theorem 6.11 we deduce that there exists $\delta > 0$ such that

$$\forall x\in I,\ \ |x - x_0| < \delta \Rightarrow g(x)\neq 0.$$

For $|h| < \delta$ such that $x_0 + h\in I$ we have

$$\left(\frac{1}{g}\right)(x_0+h) - \left(\frac{1}{g}\right)(x_0) = \frac{1}{g(x_0+h)} - \frac{1}{g(x_0)} = \frac{g(x_0) - g(x_0+h)}{g(x_0)g(x_0+h)}$$

so that

$$\frac{\left(\frac{1}{g}\right)(x_0+h)-\left(\frac{1}{g}\right)(x_0)}{h}=\frac{g(x_0)-g(x_0+h)}{h}\frac{1}{g(x_0)g(x_0+h)}$$

Hence

$$\left(\frac{1}{g}\right)'(x_0)=\lim_{h\to 0}\frac{\left(\frac{1}{g}\right)(x_0+h)-\left(\frac{1}{g}\right)(x_0)}{h}$$

$$=\lim_{h\to 0}\frac{g(x_0)-g(x_0+h)}{h}\cdot\lim_{h\to 0}\frac{1}{g(x_0)g(x_0+h)}=-\frac{g'(x_0)}{g(x_0)^2}.$$

To compute the derivative of $\frac{f}{g}$ at x_0 we use the product rule. We have

$$\frac{f}{g}=f\cdot\frac{1}{g}\Rightarrow\left(\frac{f}{g}\right)'(x_0)=\left(f\cdot\frac{1}{g}\right)'(x_0)$$

$$=f'(x_0)\frac{1}{g(x_0)}+f(x_0)\left(\frac{1}{g}\right)'(x_0)$$

$$=f'(x_0)\frac{1}{g(x_0)}-f(x_0)\frac{g'(x_0)}{g(x_0)^2}=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}.$$

□

Example 7.17. Let us see how the above rules work on some simple examples.

(a) Consider the polynomial function

$$p(x)=5-3x^2+7x^5,\quad x\in\mathbb{R}.$$

From the scalar multiplication rule and the Examples 7.10, 7.11 we deduce that each of the functions 5, $-3x^2$ and $7x^5$ is differentiable and the addition rule implies that their sum is differentiable as well. We deduce

$$p'(x)=(5)'+(-3x^2)'+(7x^5)'=-6x+35x^4.$$

(b) From the equalities

$$\frac{d}{dx}(\sin x)=\cos x,\quad \frac{d}{dx}(\cos x)=-\sin x$$

and the scalar multiplication rule we deduce that the trigonometric functions are smooth and we have

$$\frac{d^2}{dx^2}\sin x=-\sin x,\quad \frac{d^2}{dx^2}\cos x=-\cos x,$$

$$\frac{d^4}{dx^4}\sin x=\sin x,\quad \frac{d^4}{dx^4}\cos x=\cos x.$$

(c) If a is a positive real number, then

$$\log_a x=\frac{\ln x}{\ln a}$$

and we deduce

$$(\log_a x)'=\frac{1}{x\ln a}. \tag{7.13}$$

(d) If n is a natural number, then the function

$$f:\mathbb{R}\setminus\{0\}\to\mathbb{R},\ \ f(x)=\frac{1}{x^n}=x^{-n}$$

is differentiable by the quotient rule and we have

$$(x^{-n})'=\left(\frac{1}{x^n}\right)'=-\frac{nx^{n-1}}{x^{2n}}=-\frac{n}{x^{n+1}}=-nx^{-n-1}.$$

(e) From the quotient rule we deduce

$$\frac{d}{dx}\tan x=\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\frac{\cos^2 x+\sin^2 x}{\cos x^2}=\frac{1}{\cos^2 x}=1+\tan^2 x.$$

Thus

$$(\tan x)'=1+\tan^2 x=\frac{1}{\cos^2 x}. \tag{7.14}$$

(f) Using the product rule we deduce

$$\frac{d}{dx}(e^x\sin x)=e^x\sin x+e^x\cos x.$$

The above simple rules are unfortunately not powerful enough to allow us to compute the derivative of simple functions such as $e^{\sqrt{x}}$, $x>0$ or $\sqrt{2+\sin x}$. For this we need a more powerful technology. □

Theorem 7.18 (Chain Rule)**.** *Let I,J be two nontrivial intervals of the real axis. Suppose that we are given two functions $u:I\to\mathbb{R}$ and $f:J\to\mathbb{R}$ and a point $x_0\in I$ with the following properties.*

(i) *The range of the function u is contained in the interval J, i.e., $u(I)\subset J$.*

(ii) *The function u is differentiable at x_0.*

(iii) *The function f is differentiable at $u_0:=u(x_0)$.*

Then the composition

$$f\circ u:I\to\mathbb{R},\ \ f\circ u(x)=f\big(u(x)\big)$$

is differentiable at x_0 and

$$\boxed{(f\circ u)'(x_0)=f'(u_0)u'(x_0).}$$

Proof. Let us begin by giving a flawed proof. We have

$$\frac{f(u(x))-f(u(x_0))}{x-x_0}=\frac{f(u(x))-f(u(x_0))}{u(x)-u(x_0)}\cdot\frac{u(x)-u(x_0)}{x-x_0}.$$

Since u is differentiable at x_0 we have

$$\lim_{x\to x_0}u(x)=u(x_0).$$

Thus

$$\lim_{x\to x_0}\frac{f(u(x))-f(u(x_0))}{u(x)-u(x_0)}\cdot\frac{u(x)-u(x_0)}{x-x_0}$$
$$=\left(\lim_{u(x)\to u(x_0)}\frac{f(u(x))-f(u(x_0))}{u(x)-u(x_0)}\right)\cdot\left(\lim_{x\to x_0}\frac{u(x)-u(x_0)}{x-x_0}\right)$$

$$= f'(u(x_0))u'(x_0)$$

et voilà, we're done!

Unfortunately the above argument has one serious flaw. More precisely it is possible that $u(x) = u(x_0)$ for infinitely many values of x close to x_0. The quotient

$$\frac{f(u(x)) - f(u(x_0))}{u(x) - u(x_0)}$$

is ill-defined and thus the above argument is meaningless. Although problematic, the above argument displays the strategy of the proof. We need a bit of technical contortionism to avoid the problem of vanishing denominators. The details follow below.

Since f is differentiable at u_0 we deduce that it is linearly approximable at x_0. From (7.5) we deduce that

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + r(u), \quad r(u) = o(u - u_0) \text{ as } u \to u_0.$$

Recall that the equality

$$r(u) = o(u - u_0) \text{ as } u \to u_0$$

signifies that

$$\lim_{u \to u_0} \frac{r(u)}{u - u_0} = 0. \tag{7.15}$$

In particular, we deduce that

$$f\big(u(x)\big) - f\big(u(x_0)\big) = f\big(u(x)\big) - f(u_0) = f'(u_0)\big(u(x) - u(x_0)\big) + r(u(x))$$

$$\frac{f\big(u(x)\big) - f\big(u(x_0)\big)}{x - x_0} = f'(u_0)\frac{\big(u(x) - u(x_0)\big)}{x - x_0} + \frac{r(u(x))}{x - x_0}.$$

Observe that if we prove that

$$\lim_{x \to x_0} \frac{r(u(x))}{x - x_0} = 0, \tag{7.16}$$

then we deduce

$$\lim_{x \to x_0} \frac{f\big(u(x)\big) - f\big(u(x_0)\big)}{x - x_0} = f'(u_0)\lim_{x \to x_0} \frac{\big(u(x) - u(x_0)\big)}{x - x_0} = f'(u_0)u'(x_0)$$

which is the claim of the theorem.

Why do we expect (7.16) to be true? We have $r(u(x)) = o(u(x) - u_0)$, i.e., $r(u(x))$ is a tiny fraction of $u(x) - u_0$ if $u(x)$ is close to x. When x is close to x_0, then $u(x)$ is close to u_0 so $r(u(x))$ is a tiny fraction of $u(x) - u_0$ when x is close to x_0.

On the other hand, when x is close to x_0 we have

$$u(x) - u_0 = u'(x_0)(x - x_0) + o(x - x_0) = u'(x_0)(x - x_0) + \text{tiny fraction of } x - x_0$$

$$= (x - x_0)(u'(x_0) + \text{tiny number}).$$

Thus when x is close to x_0 the remainder $r(u(x))$ is a tiny fraction of $(x - x_0)(u'(x_0) + \text{tiny number})$ which in turn is obviously a tiny fraction of $(x - x_0)$. The precise proof is presented below.

To prove (7.16) it suffices to show that

$$\forall \hbar > 0 \;\; \exists d = d(\hbar) > 0: \;\; |x - x_0| < d(\hbar) \Rightarrow \frac{|r(u(x))|}{|x - x_0|} \le \hbar. \tag{7.17}$$

The function u is differentiable at x_0 and it is linearizable at this point. Hence

$$u(x) - u_0 = u'(x_0)(x - x_0) + \rho(x),\ \ \rho(x) = o(x - x_0)\ \text{ as } x \to x_0.$$

Since $\rho(x) = o(x - x_0)$ as $x \to x_0$ we deduce that there exists a small $\gamma > 0$ such that

$$|x - x_0| < \gamma \Rightarrow |\rho(x)| \leq |x - x_0|.$$

Hence, for $|x - x_0| < \gamma$ we have

$$\begin{aligned} |u(x) - u_0| &= |u'(x_0)(x - x_0) + \rho(x)| \\ &\leq |u'(x_0)||x - x_0| + |\rho(x)| \leq (|u'(x_0)| + 1)|x - x_0|. \end{aligned}$$

If we set $C := |u'(x_0)| + 1 > 0$, then we deduce

$$|x - x_0| < \gamma \Rightarrow |u(x) - u_0| \leq C|x - x_0|. \tag{7.18}$$

Note that (7.15) implies that

$$\forall \hbar > 0\ \ \exists \varepsilon(\hbar) > 0 :\ \ |u - u_0| < \varepsilon(\hbar) \Rightarrow |r(u)| \leq \hbar|u - u_0|. \tag{7.19}$$

Observe that (7.18) implies

$$|x - x_0| < \delta(\hbar) := \min\Big\{\gamma, \frac{\varepsilon(\hbar)}{c}\Big\} \Rightarrow |u(x) - u_0| \leq C|x - x_0| < \varepsilon(\hbar).$$

Using this in (7.19) we deduce that

$$|x - x_0| < \delta(\hbar) \Rightarrow |u(x) - u_0| < \varepsilon(\hbar) \stackrel{(7.19)}{\Rightarrow} |r(\,u(x)\,)| \leq \hbar|u(x) - u_0| \stackrel{(7.18)}{\leq} C\hbar|x - x_0|.$$

We have thus proved that

$$\forall \hbar > 0\ \ \exists \delta(\hbar) > 0 :\ \ |x - x_0| < \delta(\hbar) \Rightarrow \frac{|r(\,u(x)\,)|}{|x - x_0|} \leq C\hbar.$$

If we set

$$d(\hbar) := \delta(\hbar/C)$$

we obtain (7.17).

□

Remark 7.19. Since the chain rule is without a doubt the key rule in differential calculus it is perhaps appropriate to pause and provide a bit of intuition behind it. The classical point of view on this formula is in our view the most intuitive.

Before the modern concept of function (late 19th century) functions were regarded as quantities that depend on other quantities. In the chain rule we deal with three quantities denoted by x, u, f. The quantity u depends on the quantity x thus giving us the function $u = u(x)$. The quantity f depends on the quantity u thus giving us the function $f = f(u)$. Since u also depends on x, we deduce that through u as intermediary the function f also depends on x, this giving us the composition $f \circ u$.

The derivative of $f \circ u$ with respect to x measures the rate of change in the quantity f per unit of change in x. The classics would denote this rate of change by $\frac{df}{dx}$ instead of the more complete, but more cumbersome[2] $\frac{df\circ u}{dx}$. The quantity $\frac{df}{du}$ denotes the rate of change in f per unit of change in u, The quantity $\frac{du}{dx}$ is defined in a similar fashion and the chain rule takes the simpler form

$$\boxed{\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}}. \tag{7.20}$$

[2]The concept of composition of function was not clearly defined given that the concept of function was nebulous.

A less rigorous but more intuitive way of phrasing the above equality is

$$\frac{\text{change in } f}{\text{change in } x} = \frac{\text{change in } f}{\text{change in } u} \cdot \frac{\text{change in } u}{\text{change in } x}.$$

□

Let us see the chain rule at work in some simple examples.

Example 7.20. (a) Consider the function

$$\sin\sqrt{x}, \;\; x > 0.$$

It is the composition of the two functions

$$f(u) = \sin u, \;\; u(x) = \sqrt{x}.$$

Then

$$\frac{d}{dx}\sin\sqrt{x} = \frac{df}{du} \cdot \frac{du}{dx} = (\cos u) \cdot \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2\sqrt{x}}.$$

(b) Consider the function 2^x. We have

$$2^x = (e^{\ln 2})^x = e^{(\ln 2)x}.$$

It is the composition of two functions

$$f(u) = e^u, \;\; u(x) = (\ln 2)x.$$

Then

$$\frac{d}{dx}2^x = \frac{df}{du} \cdot \frac{du}{dx} = e^u(\ln 2) = e^{(\ln 2)x}\ln 2 = 2^x \ln 2.$$

More generally, if a is a positive real number then

$$\frac{d}{dx}a^x = a^x \ln a. \tag{7.21}$$

Observe that for any $\lambda \in \mathbb{R}$ we have

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x},$$

and we conclude inductively that

$$\frac{d^n}{dx^n}e^{\lambda x} = \lambda^n e^{\lambda x}, \;\; \forall n \in \mathbb{N}. \tag{7.22}$$

(c) Consider now a trickier situation. Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = x^x$. We want to prove that f is differentiable and then compute its derivative. We set

$$g(x) = \ln f(x) = x \ln x.$$

Clearly g is differentiable since it is the product of differentiable functions. From the equality

$$f(x) = e^{g(x)}$$

we deduce that f is also differentiable because it is the composition of differentiable functions. Using the chain rule we deduce

$$f'(x) = e^{g(x)}g'(x) = (x^x)g'(x) = x^x(\ln x + 1).$$

□

Theorem 7.21 (Inverse function rule). *Suppose that I, J are two intervals of the real axis and $u : I \to J$ is a bijective function satisfying the following properties.*

(i) *The function u is differentiable at the point $x_0 \in I$.*

(ii) $u'(x_0) \neq 0$.

(iii) *The inverse function u^{-1} is continuous at $y_0 = u(x_0)$.*

Then the inverse function u^{-1} is differentiable at $y_0 = u(x_0)$ and

$$(u^{-1})'(y_0) = \frac{1}{u'(x_0)}.$$

Proof. Since u is bijective we deduce that for any $y \in J$, there exists a unique $x = x(y)$ in I such that $u(x) = y$. More precisely $x(y) = u^{-1}(y)$. Since u^{-1} is continuous at y_0 we have

$$\lim_{y \to y_0} x(y) = x(y_0) = x_0.$$

Then

$$\frac{u^{-1}(y) - u^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{u(x) - u(x_0)} = \frac{1}{\frac{u(x)-u(x_0)}{x-x_0}}.$$

so that

$$\lim_{y \to y_0} \frac{u^{-1}(y) - u^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{1}{\frac{u(x)-u(x_0)}{x-x_0}} = \frac{1}{u'(x_0)}.$$

□

Example 7.22. The inverse function rule is a bit tricky to use. We discuss a few classical examples.

(a) Consider the function

$$u : (-\pi/2, \pi/2) \to (-1, 1), \quad u(x) = \sin x.$$

This function is bijective, differentiable, and the derivative $u'(x) = \cos x$ is nowhere zero. Its inverse is the continuous function

$$\arcsin : (-1, 1) \to (-\pi/2, \pi/2).$$

We have

$$\frac{d}{du} \arcsin u = \frac{1}{u'(x)} = \frac{1}{\cos x}, \quad u = \sin x.$$

Observe that on the interval $(-\pi/2, \pi/2)$ the function $\cos x$ is positive so that

$$\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - u^2}.$$

Hence

$$\frac{d}{du} \arcsin u = \frac{1}{\sqrt{1 - u^2}}, \quad \forall u \in (-1, 1). \tag{7.23}$$

A similar argument shows that

$$\frac{d}{du} \arccos u = -\frac{1}{\sqrt{1 - u^2}}, \quad \forall u \in (-1, 1). \tag{7.24}$$

(b) Consider the bijective differentiable function

$$u : (-\pi/2, \pi/2) \to \mathbb{R}, \quad u(x) = \tan x.$$

Its inverse is the function $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$. It is continuous and

$$\frac{d}{du} \arctan u = \frac{1}{u'(x)} = \frac{1}{(\tan x)'}, \quad u = \tan x.$$

Using the equality $(\tan x)' = 1 + \tan^2 x$ we deduce

$$\frac{d}{du} \arctan u = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + u^2}, \quad u = \tan x. \tag{7.25}$$

□

7.4. Fundamental properties of differentiable functions

The first fundamental result concerning differentiable functions is Fermat's Principle. Before we formulate it we need to introduce a new concept.

Definition 7.23. Suppose that $f : I \to \mathbb{R}$ is a function defined on an interval $I \subset \mathbb{R}$.

(i) A point $x_0 \in I$ is said to be a *local minimum* of f if there exists $\delta > 0$ with the following property

$$\forall x \in I, \quad |x - x_0| < \delta \Rightarrow f(x) \geq f(x_0).$$

The point x_0 is called a *strict local minimum* if there exists $\delta > 0$ with the following property

$$\forall x \in I, \quad 0 < |x - x_0| < \delta \Rightarrow f(x) > f(x_0).$$

(ii) A point $x_0 \in I$ is said to be a *local maximum* of f if there exists $\delta > 0$ with the following property

$$\forall x \in I, \quad |x - x_0| < \delta \Rightarrow f(x) \leq f(x_0).$$

The point x_0 is called a *strict local maximum* if there exists $\delta > 0$ with the following property

$$\forall x \in I, \quad 0 < |x - x_0| < \delta \Rightarrow f(x) < f(x_0).$$

(iii) A point $x_0 \in I$ is said to be a *(strict) local extremum* of f if it is either a (strict) local minimum, or a (strict) local maximum.

□

Theorem 7.24 (Fermat's Principle)**.** *Consider a function $f : [a, b] \to \mathbb{R}$ which is differentiable on the open interval (a, b). Suppose that x_0 is a local extremum of f* situated in the interior, *$x_0 \in (a, b)$. Then $f'(x_0) = 0$. In geometric terms, at an interior local extremum, the tangent line to the graph has zero slope, i.e., it is horizontal.*

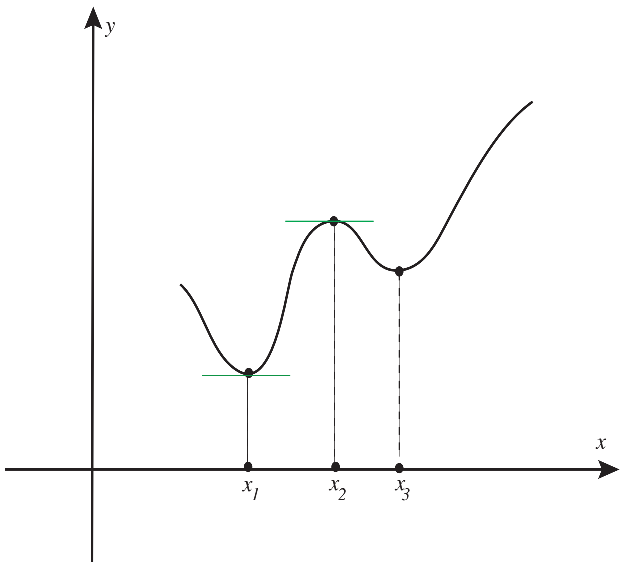

Figure 7.3. *The points x_1 and x_3 are local minima, while the point x_2 is a local maximum.*

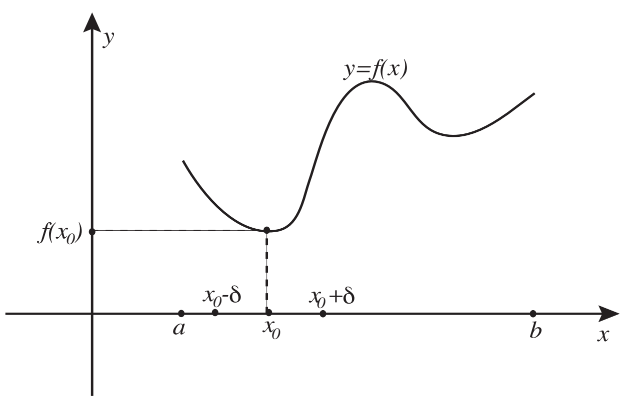

Figure 7.4. *The point x_0 is an interior local minimum.*

Proof. Assume for simplicity that x_0 is a local minimum; see Figure 7.4. Since x_0 is in the *interior* of the interval $[a, b]$ we can find $\delta > 0$ such that

$$(x_0 - \delta, x_0 + \delta) \subset (a, b) \text{ and } f(x_0) \leq f(x), \;\; \forall x \in (x_0 - \delta, x_0 + \delta).$$

We have

$$\lim_{x \searrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = \lim_{x \nearrow x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Note that

$$x \in (x_0, x_0 + \delta) \Rightarrow f(x) - f(x_0) \geq 0 \wedge x - x_0 > 0 \Rightarrow \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \Rightarrow$$

$$\Rightarrow \lim_{x \searrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \Rightarrow f'(x_0) \geq 0.$$

Similarly

$$x \in (x_0 - \delta, x_0) \Rightarrow f(x) - f(x_0) \geq 0 \wedge x - x_0 < 0 \Rightarrow \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \Rightarrow$$

$$\Rightarrow \lim_{x \nearrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \Rightarrow f'(x_0) \leq 0.$$

This proves that $f'(x_0) = 0$. □

Remark 7.25. The importance of Fermat's Principle is difficult to overestimate. Locating the local extrema of a function is a problem with a huge number of applications beyond theoretical mathematics. Fermat's Principle states that the local extrema of a differentiable function $f : [a, b] \to \mathbb{R}$ are very special points: they are either endpoints of the interval, or points where the derivative of f vanishes.

This principle reduces the search of extrema to a set much much smaller than the interval $[a, b]$. Instead of looking for the needle in a haystack, we're looking for a needle hidden in a small matchbox. There is a caveat: the matchbox could be locked and it may take some ingenuity to unlock it. □

Definition 7.26. Suppose that $f : I \to \mathbb{R}$ is a differentiable function defined on an interval $I \subset \mathbb{R}$. A point $x_0 \in I$ is called a *critical point* of f if $f'(x_0) = 0$. □

We can thus rephrase Fermat's Principle as saying that ***interior*** *local extrema must be critical points*. We want to point out that not all critical points are necessarily local extrema. For example the point $x_0 = 0$ of $f(x) = x^3$, $x \in \mathbb{R}$, is a critical point of f. However it is not a local extremum because

$$f(x) > f(0) \ \ \forall x > 0 \ \wedge \ \ f(x) < f(0) \ \ \forall x < 0.$$

Fermat's Principle has several fundamental consequences. We describe a few of them.

Theorem 7.27 (Rolle)**.** *Suppose that* $f : [a, b] \to \mathbb{R}$ *is a continuous function that is also differentiable on the open interval* (a, b)*. If* $f(a) = f(b)$*, then there exists* $\xi \in (a, b)$ *such that* $f'(\xi) = 0$.

Proof. According to Weierstrass' Theorem 6.14 there exist $x_*, x^* \in [a, b]$ such that

$$f(x_*) = \inf_{x \in [a,b]} f(x), \ \ f(x^*) = \sup_{x \in [a,b]} f(x). \tag{7.26}$$

We distinguish two cases.

1. $f(x_*) = f(x^*)$. We deduce from (7.26) that f is the constant function $f(x) = f(x_*)$, $\forall x \in [a, b]$. In particular $f'(x) = 0$, $\forall x \in (a, b)$, proving the claim in the theorem.

2. $f(x_*) < f(x^*)$. Thus x_* and x^* cannot simultaneously be endpoints of the interval $[a, b]$ because $f(a) = f(b)$. Hence at least one of the points x_* or x^* is located in the interior of the interval. Suppose for that x_* is that point. Then x_* is a local minimum of f located in the interior of (a, b). Fermat's Principle implies that $f'(x_*) = 0$. □

Theorem 7.28 (Lagrange's Mean Value Theorem)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function that is also differentiable on the open interval (a, b). Then there exists a point $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Geometrically this signifies that somewhere on the graph of f there exists a point so that the tangent to the graph at that point is parallel to the line connecting the endpoints of the graph of f; see Figure 7.5.

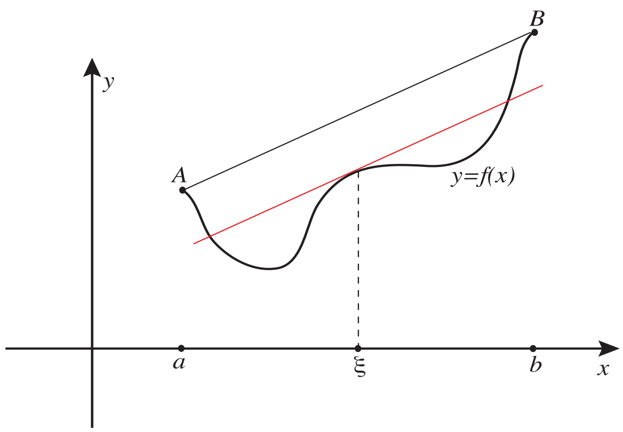

Figure 7.5. *The geometric interpretation of Theorem 7.28.*

Proof. We set

$$m := \frac{f(b) - f(a)}{b - a}.$$

The line passing through the points $A = (a, f(a))$ and $B = (b, f(b))$ has slope m and is the graph of the linear function

$$L(x) = m(x - a) + f(a).$$

Observe that

$$L(a) = f(a), \;\; L(b) = f(b), \;\; L'(x) = m, \;\; \forall x.$$

Define

$$g : [a, b] \to \mathbb{R}, \;\; g(x) = f(x) - L(x).$$

Note that g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g(a) = f(a) - L(a) = 0 = f(b) - L(b) = g(b).$$

Rolle's theorem implies that there exists $\xi \in (a, b)$ such that

$$0 = g'(\xi) = f'(\xi) - L'(\xi) = f'(\xi) - m \Rightarrow f'(\xi) = m.$$

□

Remark 7.29. In the Mean Value Theorem the requirement that f be continuous on the ***closed*** interval $[a, b]$ is essential and does not follow from the requirement that f be differentiable on the ***open*** interval (a, b).

In the theorem we have tacitly assumed that $a < b$. The result continues to be true even when $a > b$ because

$$\frac{f(b) - f(a)}{b - a} = \frac{f(a) - f(b)}{a - b}.$$

In this case ξ is a point in the open interval with endpoints a and b. □

Corollary 7.30. *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function that is also differentiable on the open interval (a, b). Then the following statements are equivalent.*

(i) *The function f is constant.*

(ii) $f'(x) = 0$, $\forall x \in (a, b)$.

Proof. The implication (i) $\Rightarrow$ (ii) is immediate since the derivative of a constant function is 0.

To prove the implication (ii) $\Rightarrow$ (i) we argue by contradiction. Suppose that there exist $x_0, x_1 \in [a, b]$, such that $x_0 < x_1$ and $f(x_0) \neq f(x_1)$. The Mean Value Theorem implies that there exists $\xi \in (x_0, x_1)$ such that

$$f'(\xi) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \neq 0.$$

□

Corollary 7.31. *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function that is differentiable on (a, b). If $f'(x) \neq 0$ for any (a, b), then f is injective.*

Proof. If $x_0, x_1 \in [a, b]$ and $x_0 \neq x_1$, say $x_0 < x_1$, then the Mean Value Theorem implies that there exists $\xi \in (x_0, x_1)$ such that

$$f(x_1) - f(x_0) = f'(\xi)(x_1 - x_0) \neq 0.$$

This proves the injectivity of f. □

Corollary 7.32. *Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function that is differentiable on (a, b). Then the following statements are equivalent.*

(i) *The function f is nondecreasing.*

(ii) $f'(x) \geq 0$, $\forall x \in (a, b)$.

Also, the following statements are equivalent.

(iii) *The function f is nonincreasing.*

(iv) $f'(x) \leq 0$, $\forall x \in (a, b)$.

Proof. (i) $\Rightarrow$ (ii). Let $x_0 \in (a, b)$. Then for $h > 0$ we have $f(x_0 + h) - f(x_0) \geq 0$ so that

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0 \Rightarrow f'(x_0) = \lim_{h \searrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

(ii) $\Rightarrow$ (i). Suppose that $x_0, x_1 \in [a,b]$ are such that $x_0 < x_1$. The Mean Value Theorem implies that there exists $\xi \in (x_0, x_1)$ such that

$$f'(\xi) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \Rightarrow f(x_1) - f(x_0) = f'(\xi)(x_1 - x_0) \geq 0.$$

□

Remark 7.33. If in the above result we replace (ii) with the stronger condition

$$f'(x) > 0, \;\; \forall x \in (a,b),$$

then we obtain a stronger conclusion namely that f is (strictly) increasing. This follows by coupling Corollary 7.32 with Corollary 7.31. □

Example 7.34. (a) We want to prove that

$$e^x \geq x + 1, \;\; \forall x \in \mathbb{R}. \tag{7.27}$$

To this aim consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - (x+1)$. This function is differentiable and $f'(x) = e^x - 1$.

We see that the derivative is positive on $(0, \infty)$ and negative on $(-\infty, 0)$. Hence f is increasing on $(0, \infty)$ and thus $f(x) > f(0) = 0$, $\forall x > 0$ and $f(x) > 0$, $\forall x \in (-\infty, 0)$. In other words,

$$e^x - (x+1) \geq 0, \;\; \forall x \in \mathbb{R},$$

which is (7.27).

(b) We want to prove that

$$x \geq \sin x, \;\; \forall x \geq 0. \tag{7.28}$$

Consider the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x - \sin x$. This function is differentiable and

$$f'(x) = 1 - \cos x \geq 0, \;\; \forall x \geq 0.$$

Hence f is nondecreasing and thus

$$x - \sin x = f(x) \geq f(0) = 0, \;\; \forall x \geq 0.$$

(c) We want to prove that

$$\cos x \geq 1 - \frac{x^2}{2}, \;\; \forall x \in \mathbb{R}. \tag{7.29}$$

Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \;\; \cos x - \left(1 - \frac{x^2}{2}\right), \;\; \forall x \in \mathbb{R}.$$

We have to prove that $f(x) \geq 0$, $\forall x \in \mathbb{R}$. We observe that f is an even function, i.e., $f(-x) = f(x)$, $\forall x \in \mathbb{R}$ so it suffices to show that $f(x) \geq 0$, $\forall x \geq 0$. Note that f is differentiable and

$$f'(x) = -\sin x + x \overset{(7.28)}{\geq} 0 \;\; \forall x \geq 0.$$

Thus f is nondecreasing on the interval $[0, \infty)$ and we conclude that $f(x) \geq f(0) = 0$, $\forall x \geq 0$.

□

Example 7.35 (Young's inequality)**.** Suppose that $p \in (1,\infty)$. Define $q \in (1,\infty)$ by $\frac{1}{p}+\frac{1}{q}=1$, i.e., $q=\frac{p}{p-1}$. Consider $f:(0,\infty)\to\mathbb{R}$

$$f(x)=x^{\alpha}-\alpha x+\alpha-1,\ \ \alpha:=\frac{1}{p}.$$

We want to prove that $f(x)\leq 0$, $\forall x>0$. We have

$$f'(x)=\alpha x^{\alpha-1}-\alpha=\alpha(x^{\alpha-1}-1)=\alpha\left(\frac{1}{x^{1-\alpha}}-1\right).$$

Observe that $f'(x)=0$ if and only if $x=1$. Moreover $f'(x)<0$ for $x>1$ and $f'(x)>0$ for $x<1$ because $1-\alpha=1-\frac{1}{p}>0$. Thus the function f increases on $(0,1)$ and decreases on $(1,\infty)$ so that

$$0=f(1)\geq f(x)\ \ \forall x>0.$$

Thus

$$x^{\alpha}-\alpha x\leq 1-\alpha=1-\frac{1}{p}>0=\frac{1}{q}.$$

If we choose $a,b>0$ and we set $x=\frac{a}{b}$ we deduce

$$\left(\frac{a}{b}\right)^{\frac{1}{p}}-\frac{1}{p}\left(\frac{a}{b}\right)^{\frac{1}{p}+\frac{1}{q}}\leq\frac{1}{q}\Rightarrow\left(\frac{a}{b}\right)^{\frac{1}{p}}\leq\frac{1}{p}\left(\frac{a}{b}\right)^{\frac{1}{p}+\frac{1}{q}}+\frac{1}{q}.$$

Multiplying both sides by $b=b^{\frac{1}{p}+\frac{1}{q}}$ we deduce

$$a^{\frac{1}{p}}b^{\frac{1}{q}}\leq\frac{a}{p}+\frac{b}{q},\ \ \forall a,b>0. \tag{7.30}$$

If we set $u:=a^{\frac{1}{p}}$, $v:=b^{\frac{1}{q}}$ then we can rewrite the above inequality in the commonly encountered form

$$uv\leq\frac{u^p}{p}+\frac{v^q}{q},\ \ \forall u,v>0,\ \ p,q>1,\ \ \frac{1}{p}+\frac{1}{q}=1. \tag{7.31}$$

The last inequality is known as *Young's inequality.* □

Corollary 7.36. *Suppose that $f:[a,b]\to\mathbb{R}$ is a continuous function that is twice differentiable on (a,b). Let $x_0\in(a,b)$ be a critical point of f, i.e., $f'(x_0)=0$. Then the following hold.*

(i) *If $f''(x_0)>0$, then x_0 is a strict local minimum of f.*

(ii) *If $f''(x_0)<0$, then x_0 is a strict local maximum of f.*

Proof. We prove only (i). Part (ii) follows by applying (i) to the new function $-f$. Suppose that

$$f'(x_0)=0,\ \ f''(x_0)>0.$$

We have to prove that there exists $\delta>0$ such that

$$0<|x-x_0|<\delta\Rightarrow f(x)>f(x_0).$$

We have

$$\lim_{x\searrow x_0}\frac{f'(x)}{x-x_0}=\lim_{x\searrow x_0}\frac{f'(x)-f'(x_0)}{x-x_0}=f''(x_0)>0.$$

Thus there exists $\delta_1>0$ such that,

$$x\in(x_0,x_0+\delta_1)\Rightarrow\frac{f'(x)}{x-x_0}>0\Rightarrow f'(x)>0.$$

The Mean Value Theorem implies that for any $x\in(x_0,x_0+\delta_1)$ there exists $\xi\in(x_0,x)$ such that

$$f(x)-f(x_0)=f'(\xi)(x-x_0).$$

Since $\xi \in (x_0, x_0+\delta_1)$ we have $f'(\xi) > 0$ and thus $f'(\xi)(x-x_0) > 0$.

Similarly

$$\lim_{x\nearrow x_0} \frac{f'(x)}{x-x_0} = \lim_{x\nearrow x_0} \frac{f'(x)-f'(x_0)}{x-x_0} = f''(x_0) > 0.$$

Thus there exists $\delta_2 > 0$ such that,

$$x \in (x_0-\delta_2, x_0) \Rightarrow \frac{f'(x)}{x-x_0} > 0 \rightarrow f'(x) < 0.$$

Hence if $x \in (x_0-\delta_2, x_0)$, then the Mean Value Theorem implies that there exists $\eta \in (x, x_0) \subset (x_0-\delta_2, x_0)$ such that

$$f(x) - f(x_0) = f'(\eta)(x-x_0) > 0.$$

If we let $\delta := \min(\delta_1, \delta_2)$, then we deduce

$$0 < |x-x_0| < \delta \Rightarrow f(x) > f(x_0).$$

□

Example 7.37. Here is a simple application of the above corollary. Fix a positive number a. Consider the function

$$f : [0,a] \to \mathbb{R}, \quad f(x) = x(a-x)^2.$$

We want to find the maximum possible value of this function. It is achieved either at one of the end points $0, a$ or at some interior point x_0. Note that $f(0) = f(a) = 0$ and $f(x) \geq 0$, $\forall x \in [0,a]$, so there must exist an interior maximum which must be a critical point. To find the critical points of f we need to solve the equation $f'(x) = 0$. We have

$$f'(x) = (a-x)^2 - 2x(a-x) = x^2 - 2ax + a^2 - 2ax + 2x^2 = 3x^2 - 4ax + a^2.$$

The discriminant of the quadratic equation $3x^2 - 4ax + a^2 = 0$ is

$$\Delta = 16a^2 - 12a^2 = 4a^2 > 0$$

Thus this quadratic equation has two roots

$$x_\pm = \frac{4a \pm 2a}{6} = a, \frac{a}{3}.$$

Only one of these roots is in the interval $(0,a)$, namely $\frac{a}{3}$. Note that

$$f''(x) = 6x - 4a, \quad f''(a/3) = 2a - 4a < 0.$$

Thus $a/3$ is the unique maximum point of f, and thus it is absolute maximum point. We have

$$f(x) \leq f(a/3) = \frac{4a^3}{27}, \quad \forall x \in [0,a].$$ □

Theorem 7.38 (Cauchy's finite increment theorem)**.** *Suppose that $f, g : [a,b] \to \mathbb{R}$ are two continuous functions that are differentiable on (a,b). Then there exists $\xi \in (a,b)$ such that*

$$f'(\xi)\big(\, g(b) - g(a)\,\big) = g'(\xi)\big(\, f(b) - f(a)\,\big). \tag{7.32}$$

In particular, if $g'(t) \neq 0$ for any $t \in (a,b)$, then $g(b) \neq g(a)$ and

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}. \tag{7.33}$$

Proof. Consider the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = f(x) \underbrace{\big(\, g(b) - g(a) \,\big)}_{=:\Delta_g} - g(x) \underbrace{\big(\, f(b) - f(a) \,\big)}_{=:\Delta_f}, \ \ \forall x \in [a, b].$$

This function is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$\begin{aligned} F(b) - F(a) &= \big(\, f(b)\Delta_g - g(b)\Delta_f \,\big) - \big(\, f(a)\Delta_g - g(a)\Delta_f \,\big) \\ &= \big(\, f(b) - f(a) \,\big)\Delta_g + \big(\, g(a) - g(b) \,\big)\Delta_f = 0. \end{aligned}$$

Rolle's theorem implies that there exists $\xi \in (a, b)$ such that $F'(\xi) = 0$. This proves (7.32). To obtain (7.33) we observe that the assumption $g'(t) \neq 0$ for any $t \in (a, b)$ implies that g is injective and thus $g(b) \neq g(a)$. Dividing both sides of (7.32) by $g(b) - g(a)$ we deduce (7.33). □

Remark 7.39. In the above theorem we have tacitly assumed that $a < b$. The result continues to be true even when $a > b$ because

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f(a) - f(b)}{g(a) - g(b)}.$$

In this case ξ is a point in the open interval with endpoints a and b. □

If $f : I \to \mathbb{R}$ is a function differentiable on the interval I, then its derivative $f' : I \to \mathbb{R}$ need not be continuous. However, the derivative is very close to being continuous in the sense that it satisfies the *intermediate value property*, just like continuous functions do.

Theorem 7.40 (Darboux)**.** *Suppose that I is an interval of the real axis and $f : I \to \mathbb{R}$ is a differentiable function. Then the derivative f' satisfies the intermediate value property: given $a, b \in I$, $a < b$, and a number γ strictly between $f'(a)$ and $f'(b)$, there exists a number $\xi \in (a, b)$ such that $f'(\xi) = \gamma$.* □

Exercise 7.25 will guide you toward a proof of this theorem which is also a consequence of Fermat's principle.

7.5. Table of derivatives

Table 7.1 summarizes the derivatives of the most frequently encountered functions.

$f(x)$	$f'(x)$
x^n, $(x \in \mathbb{R},\, n \in \mathbb{N})$	nx^{n-1}
x^{-n} $(x \neq 0,\, n \in \mathbb{N})$	$-nx^{-n-1}$
x^α, $(\alpha \in \mathbb{R},\, x > 0)$	$\alpha x^{\alpha-1}$
$\sqrt{x}$, $(x > 0)$	$\frac{1}{2\sqrt{x}}$
$\ln x$	$1/x$
e^x, $(x \in \mathbb{R})$	e^x
a^x, $(a > 0,\, x \in \mathbb{R})$	$a^x \ln a$
$\sin x$, $(x \in \mathbb{R})$	$\cos x$
$\cos x$, $(x \in \mathbb{R})$	$-\sin x$
$\tan x$, $(\cos x \neq 0)$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$
$\arcsin x$, $x \in (-1, 1)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$, $x \in (-1, 1)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$, $(x \in \mathbb{R})$	$\frac{1}{1+x^2}$
$\sinh x$, $(x \in \mathbb{R})$	$\cosh x$
$\cosh x$, $(x \in \mathbb{R})$	$\sinh x$

Table 7.1. Table of derivatives.

The *hyperbolic functions* $\sinh x$ and $\cosh x$ are defined by the equalities

$$\cosh x := \frac{e^x + e^{-x}}{2}, \;\; \sinh x = \frac{e^x - e^{-x}}{2}.$$

The function sinh is called the *hyperbolic sine* while the function cosh is called the *hyperbolic cosine*.

7.6. Exercises

Exercise 7.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$.

(i) Sketch the graph of f.
(ii) Show that f is not differentiable at 0.
(iii) Show that f is differentiable at any point $x_0 \neq 0$ and then compute the derivative of f at x_0.

□

Exercise 7.2. Prove Proposition 7.7. □

Exercise 7.3. Imitate the strategy in Example 7.15 to prove

$$\lim_{h\to 0} \frac{\cos(x_0 + h) - \cos x_0}{h} = -\sin x_0.$$

Hint. You need to use the trigonometric identities (5.33a) and (5.33c). □

Exercise 7.4. [3] Fix a natural number n and real numbers p, q.

(a) Prove that for any $t \in \mathbb{R}$ we have

$$np(tp + q)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} t^{k-1}p^k q^{n-k},$$

$$n(n-1)p^2(tp + q)^{n-2} = \sum_{k=2}^{n} k(k-1)\binom{n}{k} t^{k-2}p^k q^{n-k}.$$

Hint. Consider the function

$$f_n : \mathbb{R} \to \mathbb{R},\;\; f_n(t) = \big(\, tp + q\,\big)^n.$$

Compute the derivatives $f_n'(t)$, $f_n''(t)$. Then describe $f_n(t)$ using Newton's binomial formula and compute the same derivatives using the new description of $f_n(t)$.

(b) For any integer k, $0 \leq k \leq n$, and any $x \in \mathbb{R}$ set $w_k(x) := \binom{n}{k}x^k(1-x)^{n-k}$. Use part (a) to prove that for any $x \in \mathbb{R}$

$$1 = \sum_{k=0}^{n} w_k(x) \tag{7.34a}$$

$$nx = \sum_{k=0}^{n} k w_k(x), \tag{7.34b}$$

$$n(n-1)x^2 = \sum_{k=0}^{n} k(k-1)w_k(x) = \sum_{k=0}^{n} k^2 w_k(x) - nx, \tag{7.34c}$$

$$nx(1-x) = \sum_{k=0}^{n} (k - nx)^2 w_k(x). \tag{7.34d}$$

Hint. Use the results in (a) in the special case $p = x$, $q = 1 - x$. □

[3] The results in this exercise are very useful in probability theory.

Exercise 7.5. Suppose that the functions $f, g : I \to \mathbb{R}$ are n-times differentiable. Prove that their product $f \cdot g$ is also n-times differentiable and satisfies the generalized product rule

$$\frac{d^n}{dx^n}(fg) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}, \tag{7.35}$$

where we defined $f^{(0)} := f$, $g^{(0)} = g$.

Hint. Argue by induction on n. At some proof you need to use the Pascal formula (3.7),

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1},$$

also used in the proof of Newton's binomial formula (3.6). □

Exercise 7.6. Let n be a natural number. A real number r is said to be a *root* of order n of a polynomial $P(x)$ if there exists a polynomial $Q(x)$ with the following properties:

- $P(x) = (x - r)^n Q(x)$, $\forall x \in \mathbb{R}$.
- $Q(r) \neq 0$.

(a) Prove that if $n > 1$ and r is a a root of $P(x)$ of order n, then r is also a root of order $(n - 1)$ of $P'(x)$.

(b) Prove that for any natural numbers $k < n$ the real numbers ± 1 are roots of order $(n - k)$ of the polynomial

$$\frac{d^k}{dx^k}(x^2 - 1)^n.$$

(c) For any natural number n we define the n-th *Legendre polynomial* to be

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n}\big(x^2 - 1\big)^n.$$

Use (7.35) to compute $P_n(\pm 1)$. □

Exercise 7.7. Consider the continuous function $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$. Show that f is not differentiable at 0. □

Exercise 7.8. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & |x| \geq 1 \\ e^{-T(x)}, & |x| < 1, \end{cases} \quad \text{where } T(x) = \frac{1}{1 - x^2}, \ \ \forall |x| < 1.$$

(a) Set

$$F_n(x) := \frac{d^n}{dx^n}\big(e^{-T(x)} \big), \ \ \forall |x| < 1.$$

Prove by induction that for any $n \in \mathbb{N}$ there exists a polynomial $P_n(x)$ and a natural number k_n such that

$$F_n(x) = P_n(x) T(x)^{k_n} e^{-T(x)}, \ \ \forall |x| < 1.$$

Hint. Observe that

$$T'(x) = 2x T(x)^2.$$

(b) Prove that f is a smooth function, i.e., infinitely many times differentiable.

Hint. Prove by induction that

$$f^{(n)}(x) = \begin{cases} 0, & |x| \geq 1, \\ F_n(x), & |x| < 1. \end{cases}$$

For the inductive step observe that for $|x| < 1$ we have

$$\frac{1}{x-1} = -(x+1)T(x),$$

$$\frac{f^{(n)}(x) - f^{(n)}(1)}{x-1} = \frac{F_n(x)}{x-1} = -(x+1)T(x)F_n(x) = (x+1)P_n(x)T(x)^{k_n+1}e^{-T(x)},$$

$$\frac{F_n(x)}{x+1} = (x-1)T(x)F_n(x) = -(x-1)P_n(x)T(x)^{k_n+1}e^{-T(x)}.$$

Then

$$\lim_{x \nearrow 1} \frac{f^{(n)}(x) - f^{(n)}(1)}{x-1} = -\lim_{x \nearrow 1} \frac{F_n(x)}{x-1} = \Big(\lim_{x \nearrow 1}(x+1)P_n(x)\Big) \cdot \Big(\lim_{x \nearrow 1} T(x)^{k_n+1}e^{-T(x)}\Big)$$
$$= -2P_n(1)\Big(\lim_{x \nearrow 1} T(x)^{k_n+1}e^{-T(x)}\Big).$$

Now observe that

$$\lim_{x \nearrow 1} T(x) = \infty.$$

Use the result in Exercise 5.11 (b) to deduce

$$\lim_{x \nearrow 1} T(x)^{k_n+1}e^{-T(x)} = 0.$$

□

Exercise 7.9. Consider the function $f : (-\pi/2, \pi/2) \to \mathbb{R}$, $f(x) = \tan x$. Write the equation of the tangent line to the graph of f at the point $(\pi/4, f(\pi/4))$ □

Exercise 7.10. Find the points extrema and the intervals on which the following functions are increasing.

(i) $f(x) = \sqrt{x} - 2\sqrt{x+2}$, $x > 0$.

(ii) $g(x) = \frac{x}{x^2+1}$, $x \in \mathbb{R}$.

□

Exercise 7.11. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b). Show that if

$$\lim_{x \to a} f'(x) = A,$$

then f is differentiable at a and $f'(a) = A$. □

Exercise 7.12. Prove that if $f : I \to \mathbb{R}$ is a differentiable function defined on an interval I, and the derivative f' is bounded on I, then f is a Lipschitz function, i.e.,

$$\exists L > 0, \ \ \forall x, y \in I \ \ |f(x) - f(y)| \leq L|x - y|. \qquad \square$$

Exercise 7.13. Use the Mean Value Theorem to prove that

$$|\sin(x) - \sin(y)| \leq |x - y|, \ \ \forall x, y \in \mathbb{R}. \qquad \square$$

Exercise 7.14. Fix a real number λ and suppose that $u:\mathbb{R}\to\mathbb{R}$ is a differentiable function satisfying the differential equation

$$u'(t)=\lambda u(t),\ \ \forall t\in\mathbb{R}.$$

Prove that there exists a constant $c\in\mathbb{R}$ such that $u(t)=ce^{\lambda t}$, $\forall t\in\mathbb{R}$.

Hint: Show that the function $f(t)=e^{-\lambda t}u(t)$, $t\in\mathbb{R}$ is constant. □

Exercise 7.15. Suppose that b,c are real numbers and $u,v:\mathbb{R}\to\mathbb{R}$ are twice differentiable functions satisfying the differential equation

$$u''(t)+bu'(t)+cu(t)=0=v''(t)+bv'(t)+cv(t),\ \ \forall t\in\mathbb{R}.$$

Define the *wronskian* to be the function

$$W(t)=u(t)v'(t)-u'(t)v(t),\ \ t\in\mathbb{R}.$$

Prove that

$$W'(t)+bW(t)=0$$

and deduce that

$$W(t)=W(0)e^{-bt}.$$

Hint. You may want to use Exercise 7.14. □

Exercise 7.16. (a) Suppose that $u:\mathbb{R}\to\mathbb{R}$ is a twice differentiable function satisfying the differential equation

$$u''(t)+u(t)=0,\ \ \forall t\in\mathbb{R}. \tag{7.36}$$

Prove that

$$u'(t)^2+u(t)^2=u'(0)^2+u(0)^2,\ \ \forall t\in\mathbb{R}.$$

(b) Suppose that $u,v:\mathbb{R}\to\mathbb{R}$ are twice differentiable functions satisfying the differential equation (7.36), i.e.,

$$u''(t)+u(t)=0=v''(t)+v(t),\ \ \forall t\in\mathbb{R}.$$

Show that the difference $w(t)=u(t)-v(t)$ also satisfies the differential equation (7.36). Use part (a) to prove that if $u(0)=v(0)$ and $u'(0)=v'(0)$, then $u(t)=v(t)$, $\forall t\in\mathbb{R}$.

(c) Can you think of a function $u:\mathbb{R}\to\mathbb{R}$ satisfying (7.36) and the initial conditions

$$u(0)=0,\ \ u'(0)=1?$$

□

Exercise 7.17. (a) Prove that for any real number $\alpha\geq 1$ and any $x>-1$ we have

$$(1+x)^\alpha\geq 1+\alpha x.$$

(b) Prove by induction that for any natural number n and any $x\geq 0$ we have

$$1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}\leq e^x.$$

Hint. Have a look at Example 7.34. □

Exercise 7.18. Prove that

$$\sin x\geq x-\frac{x^3}{6},\ \ \forall x\geq 0.$$

Hint. Have a look at Example 7.34. □

Exercise 7.19. Prove that for any $x > 0$ we have

$$\ln(x+1) - \ln x < \frac{1}{x}.$$

Conclude that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} > \ln(n+1), \quad \forall n \in \mathbb{N}.$$ □

Exercise 7.20. Fix a real number $s \in (0,1)$. Prove that for any $x > 0$ we have

$$(1+x)^{1-s} - x^{1-s} < \frac{1-s}{x^s}.$$

Conclude that

$$1 + \frac{1}{2^s} + \cdots + \frac{1}{n^s} > \frac{1}{1-s}\big((n+1)^{1-s} - 1\big).$$ □

Exercise 7.21. Find the maximum possible volume of an open rectangular box that can be obtained from a square sheet of cardboard with a 6 ft side by cutting squares at each of the corners and bending up the ends of the resulting cross-like figure; see Figure 7.6. □

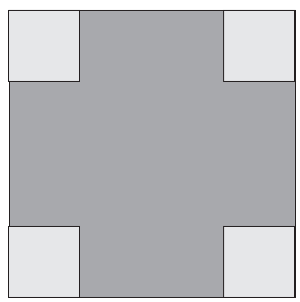

Figure 7.6. *Cutting out a box.*

Exercise 7.22. Prove that among all the rectangles with given perimeter P the square has the largest area. □

Exercise 7.23. Suppose that $f : [-1,1] \to \mathbb{R}$ is a differentiable function.

(a) Prove that if f is even, i.e., $f(x) = f(-x)$, $\forall x \in [-1,1]$, then $f'(x)$ is odd, $f'(-x) = -f'(x)$, $\forall x \in [-1,1]$. In particular, $f'(0) = 0$.

(b) Prove that if f is odd, then f' is even. □

Exercise 7.24. Fix a natural number n and suppose that $f : (a,b) \to \mathbb{R}$ is a $2n$-times differentiable function. Prove the following statements.

(a) If $x_0 \in (a,b)$ satisfies

$$f'(x_0) = \cdots = f^{(2n-1)}(x_0) = 0, \quad f^{(2n)}(x_0) > 0,$$

then x_0 is a strict local minimum of f.

(a) If $x_0 \in (a,b)$ satisfies

$$f'(x_0) = \cdots = f^{(2n-1)}(x_0) = 0, \quad f^{(2n)}(x_0) < 0,$$

then x_0 is a strict local maximum of f.

Hint. Use proof of Corollary 7.36 as inspiration and prove (in case (a)) that there exists $\delta > 0$ such that for $x \in (x_0, x_0 + \delta)$ we have $f^{(k)}(x) > 0, \forall k = 1, \dots, 2n-1$ and for $x \in (x_0 - \delta, x_0)$ we have $f^{(k)}(x) < 0$, $\forall k = 1, \dots, 2n-1$. □

Exercise 7.25 (Intermediate value property of derivatives)**.** Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable function.

(a) Prove that if $f'(a) < 0 < f'(b)$, then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Hint. Prove that the global minimum of f whose existence is guaranteed by Corollary 6.16 is located in the *interior* of the interval $[a, b]$.

(b) More generally, prove that if $f'(a) < f'(b)$ and $m \in (f'(a), f'(b))$, then there exists $\xi \in (a, b)$ such that $f'(\xi) = m$.

Hint. Use part (a) for the new function $g(x) = f(x) - mx$. □

7.7. Exercises for extra-credit

Exercise* 7.1. Suppose $f_n : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of differentiable functions functions with the following properties.

(i) The sequence of derivatives $f_n' : [a, b] \to \mathbb{R}$ converge that converges *uniformly* to a function $g : [a, b] \to \mathbb{R}$.

(ii) The sequence $f_n : [a, b] \to \mathbb{R}$ converges *pointwisely* to a function $f : [a, b] \to \mathbb{R}$.

Prove that the following hold.

(a) The sequence $f_n : [a, b] \to \mathbb{R}$ converges *uniformly* to $f : [a, b] \to \mathbb{R}$.

(b) The function f is differentiable and $f' = g$, i.e., the sequence $f_n' : [a, b] \to \mathbb{R}$ converges uniformly to f'.

Hint. Use Exercise 6.14 and the Mean Value Theorem. □

Exercise* 7.2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{h \searrow 0} \frac{f(x + 2h) - f(x + h)}{h} = 0, \ \ \forall x \in \mathbb{R}.$$

Prove that f is a constant function.

Hint. Argue by contradiction and assume there exist a, b such that $f(a) \neq f(b)$, say $f(a) < f(b)$. Consider the function $g(x) = f(x) - mx$, $m := \frac{f(b)-f(a)}{b-a}$. Note that $g(a) = g(b)$ and

$$\lim_{h \searrow 0} \frac{g(x + 2h) - g(x + h)}{h} = -m < 0,$$

and prove that g admits a local maximum in $[a, b)$. □

Exercise* 7.3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a C^2-function, i.e., twice differentiable and the second derivative is continuous. Show that if the functions f and $f^{(2)}$ are bounded on $\mathbb{R}$, then so is the function f'. □

Exercise* 7.4 (Bernstein)**.** Let $f:[0,1]\to\mathbb{R}$ be a continuous function. For any $n\in\mathbb{N}$ we denote by $B_n^f(x)$ the n-th Bernstein polynomial determined by f,

$$B_n(x)=\sum_{k=0}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k}.$$

(a) Show that for any $x\in[0,1]$ we have

$$f(x)-B_n^f(x)=\sum_{k=0}^{n}\big(\, f(x)-f(k/n)\,\big)\binom{n}{k}x^k(1-x)^k.$$

(b) Show that for any $\delta\in(0,1)$ and $x\in[0,1]$ we have

$$\sum_{|k/n-x|\geq\delta}\binom{n}{k}x^k(1-x)^k\leq\sum_{k=0}\frac{(k-nx)^2}{n^2\delta^2}\leq\frac{x(1-x)}{n\delta^2}.$$

(c) Use (a) and (b) to prove that as $n\to\infty$ the sequence $(B_n^f(x))$ converges to $f(x)$ uniformly in $x\in[0,1]$.

Hint. Use the equalities in Exercise 7.4. □

Chapter 8

Applications of differential calculus

8.1. Taylor approximations

The concept of derivative is based on the idea of approximation. Thus, if $f : I \to \mathbb{R}$ is a differentiable function and $x_0 \in I$, then the linearization of f at x_0,

$$L(x) = f(x_0) + f'(x_0)(x - x_0),$$

is a good approximation for $f(x)$ when x is not too far from x_0. More precisely, the error

$$r(x) = f(x) - L(x)$$

is $o(x - x_0)$, much much smaller than $|x - x_0|$, which itself is small when x is close to x_0. In this section we want to refine and improve this observation.

Definition 8.1. Suppose that $f : I \to \mathbb{R}$ is an n-times differentiable function defined on an interval I. For $x_0 \in I$ we define the *degree n Taylor polynomial* of f at x_0 to be

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Often the Taylor polynomial of f at $x_0 = 0$ is referred to as the *MacLaurin polynomial*.

If $f : I \to \mathbb{R}$ is a smooth function, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is called the *Taylor series* of the smooth function f at the point x_0. □

Example 8.2. (a) Consider a differentiable function $f : I \to \mathbb{R}$. Then the degree 1 Taylor polynomial of f at x_0 is

$$T_1(x) = f(x_0) + f'(x_0)(x - x_0).$$

Thus, $T_1(x)$ is the linearization of f at x_0.

(b) Consider the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^x$. We know that $f^{(n)}(x)=e^x$, $\forall n\in\mathbb{N}$, $x\in\mathbb{R}$ and we deduce that

$$f^{(k)}(0)=1,\ \ \forall k\in\mathbb{N}.$$

In particular, the degree n Taylor polynomial of e^x at $x_0=0$ is

$$T_n(x)=1+\frac{x}{1!}+\cdots+\frac{x^n}{n!}.$$

The Taylor series of e^x at $x_0=0$ is

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

(c) Consider the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin x$. We have

$$f^{(4k)}(x)=\sin x,\ \ f^{(4k+1)}(x)=\cos x,\ \ f^{(4k+2)}(x)=-\sin x,\ \ f^{(4k+3)}(x)=-\cos x,\ \ \forall k\geq 0,$$

$$f^{(4k)}(0)=0,\ \ f^{(4k+1)}(0)=1,\ \ f^{(4k+2)}(0)=0,\ \ f^{(4k+3)}(0)=-1.$$

We deduce that the Taylor polynomials of $\sin x$ at $x_0=0$ are

$$T_1(x)=f(0)+\frac{f'(0)}{1!}x=x,$$

$$T_2(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2=x,$$

$$T_3(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3=x-\frac{x^3}{6},$$

$$T_n(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots.$$

The Taylor series of $\sin x$ at $x_0=0$ is

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

(d) Consider the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\cos x$. We have

$$f^{(4k)}(x)=\cos x,\ \ f^{(4k+1)}(x)=-\sin x,\ \ f^{(4k+2)}(x)=-\cos x,\ \ f^{(4k+3)}(x)=\sin x,\ \ \forall k\geq 0$$

$$f^{(4k)}(0)=1,\ \ f^{(4k+1)}(0)=0,\ \ f^{(4k+2)}(0)=-1,\ \ f^{(4k+3)}(0)=0.$$

We deduce that the Taylor polynomials of $\cos x$ at $x_0=0$ are

$$T_1(x)=f(0)+\frac{f'(0)}{1!}x=1,$$

$$T_2(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2=1-\frac{x^2}{2!},$$

$$T_3(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3=1-\frac{x^2}{2!},$$

$$T_n(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots.$$

The Taylor series of $\cos x$ at $x_0=0$ is

$$\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

(e) Fix a real number α and define $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^\alpha$. We have

$$f'(x) = \alpha x^{\alpha-1},\;\; f^{(2)}(x) = \alpha(\alpha-1)x^{\alpha-2},\;\; f^{(k)}(x) = \alpha(\alpha-1)\cdots(\alpha-(k-1)\,)x^{\alpha-k}.$$

We deduce that

$$f^{(k)}(1) = \alpha(\alpha-1)\cdots(\alpha-(k-1)\,)$$

and thus the degree n Taylor polynomial of x^α at $x_0 = 1$ is

$$T_n(x) = 1 + \frac{\alpha}{1!}(x-1) + \frac{\alpha(\alpha-1)}{2!}(x-1)^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1)\,)}{n!}(x-1)^n.$$

The coefficients of the above polynomial coincide with the binomial coefficients if α is a natural number. For this reason, for any $\alpha \in \mathbb{R}$ we introduce the notation

$$\binom{\alpha}{0} = 1,\;\; \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1)\,)}{n!},\;\; n \in \mathbb{N}.$$

The degree n Taylor polynomial of x^α at $x_0 = 1$ can then be described in the more compact form

$$T_n(x) = \sum_{k=0}^{n} \binom{\alpha}{k}(x-1)^{\alpha-k}.$$ □

Remark 8.3. The degree n Taylor polynomial of a function f at a point x_0 is the unique polynomial of degree $\leq n$ such that

$$T_n(x_0) = f(x_0),\;\; T_n'(x_0) = f'(x_0),\;\; T_n^{(k)}(x_0) = f^{(k)}(x_0),\;\; \forall k = 1,\dots,n.$$

Exercise 8.1 asks you to prove this fact. □

Example 8.2 shows that the degree 1 Taylor polynomial of a differentiable function at a point x_0 is the linear approximation of f at x_0, and we know that it provides a very good approximation for $f(x)$ if x is near x_0. The next result states that the same is true for the higher degree Taylor polynomials.

Theorem 8.4 (Taylor approximation)**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is $(n+1)$-times differentiable, $n \in \mathbb{N}$. Fix $x_0 \in [a,b]$. We form the degree n Taylor polynomial of f at x_0*

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

and we consider the remainder (or error)

$$R_n(x_0,x) = f(x) - T_n(x),\;\; x \in [a,b].$$

Fix $x \in [a,b]$, $x \neq x_0$, and a continuous function $\varphi : [x_0,x] \to \mathbb{R}$ which is differentiable on (x_0,x) and $\varphi'(t) \neq 0$, $\forall t \in (x_0,x)$. (Here we are deliberately a bit negligent and we think of $[x_0,x]$ as the closed interval with endpoints x_0,x, even in the case $x_0 > x$.)

Then there exists ξ in the open interval with endpoints x_0 and x such that

$$R_n(x_0,x) = \frac{\varphi(x)-\varphi(x_0)}{n!\varphi'(\xi)} f^{(n+1)}(\xi)(x-\xi)^n. \tag{8.1}$$

Proof. Consider the function $F : [x_0, x] \to \mathbb{R}$ given by

$$F(t) = f(x) - \left(f(t) + \frac{f'(t)}{1!}(x-t) + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n \right), \;\; \forall t \in [x_0, x].$$

Note that $F(x) = 0$, $F(x_0) = R_n(x_0, x)$. From the Cauchy mean value theorem (7.33) (and Remark 7.39) we deduce that there exists ξ in the interval (x_0, x) such that

$$\frac{F(x) - F(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{F'(\xi)}{\varphi'(\xi)}.$$

Now observe that

$$\begin{aligned} -F'(t) &= f'(t) + \left(\frac{f''(t)}{1!}(x-t) - \frac{f'(t)}{1!} \right) + \left(\frac{f^{(3)}(t)}{2!}(x-t)^2 - \frac{f^{(2)}(t)}{1!}(x-t) \right) \\ &\quad + \cdots + \left(\frac{f^{(n+1)}(t)}{n!}(x-t)^n - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} \right) \\ &= \frac{f^{(n+1)}(t)}{n!}(x-t)^n. \end{aligned}$$

Thus

$$-\frac{R_n(x_0, x)}{\varphi(x) - \varphi(x_0)} = \frac{F(x) - F(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{F'(\xi)}{\varphi'(\xi)} = -\frac{f^{(n+1)}(t)(x-\xi)^n}{n!\varphi'(\xi)}$$

The last equality clearly implies (8.1). □

If we let $\varphi(t) = (x-t)^{n+1}$ in the above theorem, we obtain the following important consequence.

Corollary 8.5 (Lagrange remainder formula)**.** *There exists* $\xi \in (x_0, x)$ *such that*

$$\boxed{f(x) - T_n(x) = R_n(x_0, x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-x_0)^{n+1}.} \tag{8.2}$$

Proof. We have $\varphi(x) = 0$ and $\varphi(x) - \varphi(x_0) = -(x-x_0)^{n+1}$, $\varphi'(\xi) = -(n+1)(x-\xi)^n$. □

Remark 8.6. Let us explain how this works in applications. Suppose that $f : [a, b] \to \mathbb{R}$ is $(n+1)$-times differentiable and $x_0 \in [a, b]$. The degree n Taylor polynomial of f at x_0 is

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

It is convenient to introduce the notation $h = x - x_0$ so that $x = x_0 + h$ and we deduce

$$T_n(x_0 + h) = f(x_0) + \frac{f'(x_0)}{1!}h + \cdots + \frac{f^{(n)}(x_0)}{n!}h^n.$$

If h is sufficiently small, then $T_n(x_0 + h)$ is an approximation for $f(x_0 + h)$. The error of this approximation is given by the remainder $R_n(x_0, x) = f(x_0 + h) - T_n(x_0 + h)$. This remainder really depends only on the difference $h = x - x_0$ and, to emphasize this fact, we will write $R_n(x_0, h)$ instead of $R_n(x_0, x)$ in the argument below. Also, for simplicity, we will denote by $(x_0, x_0 + h)$ the open interval with endpoints x_0 and $x_0 + h$. (Note that $x_0 + h < x_0$ when $h < 0$.)

The Lagrange remainder formula tells us that there exists $\xi \in (x_0, x_0 + h)$

$$R_n(x_0, h) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) h^{n+1},$$

If we define

$$M_{n+1}(x_0, h) := \sup_{\xi \in [x_0, x_0+h]} |f^{(n+1)}(\xi)|,$$

then we deduce

$$|\, R_n(x_0, h)\,| \leq \frac{M_{n+1}(x_0, h)|h|^{n+1}}{(n+1)!}. \tag{8.3}$$

If the right-hand-side of the above inequality is small, then the error has to be small. The above result implies that

$$\big|\, f(x) - T_n(x)\,\big| = O\big(\, |x - x_0|^{n+1}\,\big) \text{ as } x \to x_0, , \tag{8.4}$$

where O is Landau's symbol defined in (5.34). □

Example 8.7. Let us show how the above remark works in a rather concrete case. Suppose $f(x) = \sin x$. We use Taylor approximations of $\sin x$ at $x_0 = 0$. For example, the degree 4 Taylor polynomial of $\sin x$ at $x_0 = 0$ is

$$T_4(h) = \sin(0) + \frac{\cos(0)}{1!} h - \frac{\sin(0)}{2!} h^2 - \frac{\cos(0)}{3!} h^3 + \frac{\sin(0)}{4!} h^4 = h - \frac{h^3}{3!} = h - \frac{h^3}{6}.$$

We have

$$\sin h \approx h - \frac{h^3}{6}.$$

To estimate the error of this approximation we use (8.2). The 5th derivative of $\sin x$ is $\cos x$ so that $|\cos \xi| \leq 1$, $\forall x \in \mathbb{R}$. We deduce from (8.2) that for some ξ between 0 and x we have

$$\left|\sin h - \left(h - \frac{h^3}{6}\right)\right| = \frac{|\cos \xi|}{5!} h^5 \leq \frac{|h|^5}{5!} = \frac{|h|^5}{120}.$$

If for example $|h| \leq \frac{1}{2}$, then

$$\frac{|h|^5}{120} \leq \frac{1}{32 \cdot 120} = \frac{1}{3840} < \frac{1}{10^3}.$$

Thus for $|h| \leq \frac{1}{2}$ the expression $h - \frac{h^3}{6}$ approximates $\sin h$ up to two decimals. For example

$$0.5 - (0.5)^3/6 = 0.47916... \Rightarrow \sin 0.5 = 0.47...$$

If $h = \frac{1}{4}$, then

$$\frac{|h|^5}{120} = \frac{1}{4^5 \cdot 120} = \frac{1}{1024 \cdot 120} = \frac{1}{122880} \leq \frac{1}{10^5},$$

and $0.25 - (0.25)^3/6$ computes $\sin(0.25)$ up to four decimals. Thus

$$0.25 - (0.25)^3/6 = 0.248666... \Rightarrow \sin(0.25) = 0.2486....$$

In Figure 8.1 we have depicted side-by-side the graph of $\sin(x)$ for $|x| \leq 10$ and the graph of $T_7(x)$, its degree 7 Taylor approximation at $x_0 = 0$. While $T_7(x)$ takes large values for $|x|$ large, it matches very well the graph of $\sin x$ on the interval $[-3, 3]$. □

Here is a nice consequence of Corollary 8.5.

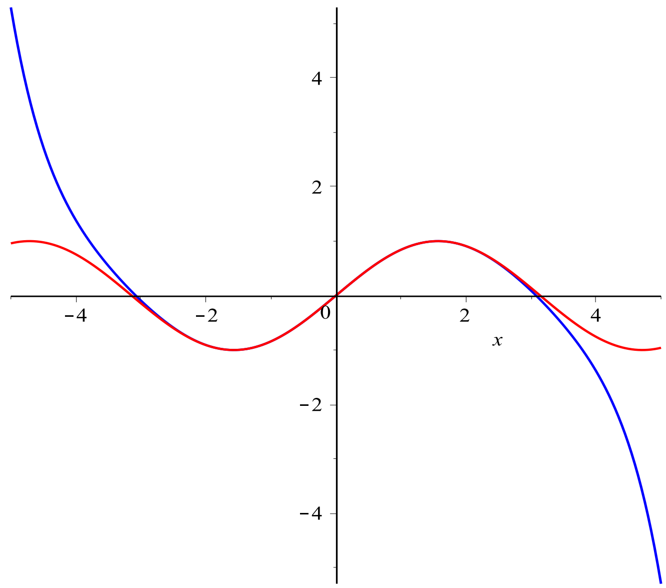

Figure 8.1. *The graphs of* $\sin x$ *and its* *degree 7 Taylor approximation* *at the origin.*

Corollary 8.8. *For any* $x \in \mathbb{R}$ *we have*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{8.5}$$

Note that for $x = 1$ *the above equality specializes to (4.26).*

Proof. Observe that for any natural number n the partial sum

$$s_n(x) = 1 + \frac{x}{1!} + \cdots + \frac{x^n}{n!}$$

is the n-th Taylor polynomial of e^x at $x_0 = 0$. Corollary 8.5 implies that there exists a real number ξ_n between 0 and x such that

$$e^x - s_n(x) = e^{\xi_n} \frac{x^{n+1}}{(n+1)!}.$$

Observe that since $-|x| \leq \xi_n \leq |x|$ we have $e^{\xi_n} \leq e^{|x|}$ so that

$$\big|\, e^x - s_n(x) \,\big| \leq e^{|x|} \frac{|x|^{n+1}}{(n+1)!}. \tag{8.6}$$

From (4.8) we deduce that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

The Squeezing Principle then implies that

$$\lim_{n\to\infty} \big|\, e^x - s_n(x) \,\big| = 0.$$

$\square$

Remark 8.9. The above proof shows a bit more namely that for any $R > 0$, the partial sums $s_n(x)$ converge to e^x *uniformly* on $[-R, R]$. Indeed, if $x \in [-R, R]$ so that $|x| \leq R$, then (8.6) implies that

$$\left| e^x - s_n(x) \right| \leq e^R \frac{R^{n+1}}{(n+1)!}, \ \ \forall |x| \leq R.$$

Note that the right-hand-side of the above inequality is independent of x and converges to 0 as $n \to \infty$ according to (4.8). Weierstrass criterion in Exercise 6.6 implies the claimed uniform convergence. □

8.2. L'Hôpital's rule

Differential calculus is also very useful in dealing with singular limits such as $\frac{0}{0}$, $\frac{\infty}{\infty}$.

Proposition 8.10 (L'Hôpital's Rule)**.** *Let $a, b \in [-\infty, \infty]$, $a < b$. Suppose that the differentiable functions $f, g : (a, b) \to \mathbb{R}$ satisfy the following conditions.*

(i) $g'(x) \neq 0$, $\forall x \in (a, b)$.

(ii)
$$\lim_{x \nearrow b} \frac{f'(x)}{g'(x)} = A \in [-\infty, \infty].$$

(iii) *Either*
$$\lim_{x \nearrow b} f(x) = \lim_{x \nearrow b} g(x) = 0, \tag{iii$_0$}$$
or
$$\lim_{x \nearrow b} g(x) = \pm\infty. \tag{iii$_\infty$}$$

Then
$$\lim_{x \nearrow b} \frac{f(x)}{g(x)} = A.$$

Proof. Let us first observe that (i) and Rolle's Theorem imply that g is injective. Hence, there exists $a' \in [a, b)$ such that $g(x) \neq 0$, $\forall x \in (a', b)$. Without any loss of generality we can assume that $a = a'$ since we are interested in the behavior of f, g near b. We have to prove that for any sequence $x_n \in (a, b)$ such that $\lim x_n = b$ we have

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = A.$$

Fix one such sequence $(x_n)_{n \in \mathbb{N}}$. At this point we want to invoke the following auxiliary fact whose proof we postpone.

Lemma 8.11. *There exists a sequence (y_n) in (a, b) such that $x_n \neq y_n$, $\forall n$, $\lim_{n \to \infty} y_n = b$ and*

$$\lim \left(\frac{|f(y_n)| + |g(y_n)|}{|g(x_n)|} \right) = 0.$$ □

Choose a sequence (y_n) as in the above lemma so that

$$\lim \frac{f(y_n)}{g(x_n)} = \lim \frac{g(y_n)}{g(x_n)} = 0.$$

From Cauchy's Finite Increment Theorem 7.38 we deduce that there exists $\xi_n \in (x_n, y_n)$ such that

$$\frac{f(x_n) - f(y_n)}{g(x_n) - g(y_n)} = \frac{f'(\xi_n)}{g'(\xi_n)}.$$

Since $x_n \to b$ we deduce $\xi_n \to b$ so that

$$\lim_{n\to\infty} \frac{f(x_n) - f(y_n)}{g(x_n) - g(y_n)} = \lim_{n\to\infty} \frac{f'(\xi_n)}{g'(\xi_n)} = A. \tag{8.7}$$

On the other hand, for any n we have

$$\frac{f(x_n) - f(y_n)}{g(x_n) - g(y_n)} = \frac{f(x_n) - f(y_n)}{g(x_n)\big(1 - \frac{g(y_n)}{g(x_n)}\big)} = \frac{\frac{f(x_n)}{g(x_n)} - \frac{f(y_n)}{g(x_n)}}{1 - \frac{g(y_n)}{g(x_n)}}.$$

We deduce

$$\frac{f(x_n)}{g(x_n)} - \frac{f(y_n)}{g(x_n)} = \left(1 - \frac{g(y_n)}{g(x_n)}\right) \frac{f(x_n) - f(y_n)}{g(x_n) - g(y_n)},$$

so that,

$$\begin{aligned} \frac{f(x_n)}{g(x_n)} &= \frac{f(y_n)}{g(x_n)} + \left(1 - \frac{g(y_n)}{g(x_n)}\right) \frac{f(x_n) - f(y_n)}{g(x_n) - g(y_n)} \\ &= \underbrace{\frac{f(y_n)}{g(x_n)}}_{\to 0} + \underbrace{\left(1 - \frac{g(y_n)}{g(x_n)}\right)}_{\to 1} \cdot \underbrace{\frac{f'(\xi_n)}{g'(\xi_n)}}_{\to A} \to A. \end{aligned}$$

All there is left to do is prove Lemma 8.11.

Proof of Lemma 8.11 We consider two cases.

1. Suppose that (iii_0) holds, i.e.,

$$\lim_{x\nearrow b} f(x) = \lim_{\nearrow b} g(x) = 0.$$

Then for any n we can find $y_n \in (x_n, b)$ such that

$$|f(y_n)| + |g(y_n)| < \frac{1}{n}|g(x_n)|.$$

so that

$$\frac{|f(y_n)| + |g(y_n)|}{|g(x_n)|} < \frac{1}{n}, \quad \forall n,$$

and thus

$$\lim_{n\to\infty} \frac{|f(y_n)| + |g(y_n)|}{|g(x_n)|} = 0.$$

---

2. Suppose that (iii_∞) holds, i.e.,

$$\lim_{n\to\infty} g(x_n) = \pm\infty.$$

For $t \in (a, b)$ we set $h(t) := |f(t)| + |g(t)|$. We construct inductively an increasing sequence of natural numbers (n_k) as follows.

A. Since $|g(x_n)| \to \infty$ there exists $n_0 \in \mathbb{N}$ such that

$$|g(x_n)| > h(x_1), \quad \forall n \geq n_0.$$

B. Since $|g(x_n)| \to \infty$, for any $k \in \mathbb{N}$, $k > 1$, we can find $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and

$$|g(x_n)| > 2^k h(x_{n_{k-1}}) \;\; \forall n \geq n_k. \tag{8.8}$$

Now define y_n by setting

$$y_n := \begin{cases} x_1, & 1 \leq n < n_1 \\ x_{n_{k-1}}, & n_k \leq n < n_{k+1},\ k \in \mathbb{N}. \end{cases}$$

Observe that for $n \in [n_k, n_{k+1})$ we have

$$\frac{h(y_n)}{|g(x_n)|} = \frac{|h(x_{n_{k-1}})|}{g(x_n)} \stackrel{(8.8)}{<} \frac{1}{2^k}.$$

This proves that

$$\lim_{n\to\infty} \frac{h(y_n)}{|g(x_n)|} = 0.$$ □

□

Remark 8.12. Proposition 8.10 has a counterpart involving the left limit $\lim_{x\searrow a}$. Its statement is obtained from the statement of Proposition 8.10 by globally replacing the limit at b with the limit at a. The proof is entirely similar. □

Example 8.13. (a) We want to compute

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}.$$

According to L'Hôpital's theorem we have

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{(1-\cos x)'}{(x^2)'} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}.$$

(b) Consider the function $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^x$. We want to investigate the limit

$$\lim_{x\to 0} x^x.$$

Formally the limit ought to be 0^0, but we do not know what 0^0 means. Consider a new function

$$g(x) = \ln x^x = x \ln x, \quad x > 0.$$

In this case we have

$$\lim_{x\to 0+} g(x) = 0 \cdot (-\infty)$$

which is a degenerate limit. We rewrite

$$g(x) = \frac{\ln x}{\frac{1}{x}}$$

and we observe that in this case

$$\lim_{x\to 0+} g(x) = -\frac{\infty}{\infty}$$

which suggests trying L'Hôpital's rule. We have

$$(\ln x)' = \frac{1}{x}, \quad (1/x)' = -\frac{1}{x^2}$$

and

$$\frac{1/x}{-1/x^2} = -x \to 0 \text{ as } x \to 0+.$$

Hence

$$\lim_{x\to 0+} g(x) = 0 \Rightarrow \lim_{x\to 0+} f(x) = e^0 = 1.$$ □

8.3. Convexity

We begin with a simple geometric observation.

Proposition 8.14. *Let $x, x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$. The following statements are equivalent.*

(i) $x \in [x_1, x_2]$.

(ii) *There exist $t_1, t_2 \geq 0$ such that $t_1 + t_2 = 1$ and $x = t_1 x_1 + t_2 x_2$.*

Proof. (i) $\Rightarrow$ (ii) Suppose $x \in [x_1, x_2]$. We set

$$t_1 := \frac{x_2 - x}{x_2 - x_1}, \;\; t_2 := \frac{x - x_1}{x_2 - x_1}. \tag{8.9}$$

Since $x_1 \leq x \leq x_2$ we deduce that $t_1, t_2 \geq 0$. We observe that

$$t_1 + t_2 = \frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1,$$

and

$$t_1 x_1 + t_2 x_2 = \frac{x_1(x_2 - x) + x_2(x - x_1)}{x_2 - x_1} = \frac{x_2 x - x_1 x}{x_2 - x_1} = x. \tag{8.10}$$

(ii) $\Rightarrow$ (i) Suppose that there exist $t_1, t_2 \geq 0$ such that $t_1 + t_2 = 1$ and $x = t_1 x_1 + t_2 x_2$. We have

$$\begin{aligned} x - x_1 &= (t_1 - 1)x_1 + t_2 x_2 = -t_2 x_1 + t_2 x_2 = t_2(x_2 - x_1) \geq 0, \\ x_2 - x &= (1 - t_2)x_2 - t_1 x_1 = t_1 x_2 - t_1 x_1 = t_1(x_2 - x_1) \geq 0. \end{aligned}$$

Hence $x_1 \leq x \leq x_2$. □

Given a function $f : (a, b) \to \mathbb{R}$ and $x_1, x_2 \in (a, b)$, $x_1 < x_2$, we denote by $L^f_{x_1, x_2}$ the linear function whose graph contains the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f. The slope of this line is

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

and thus the equation of this line is

$$\begin{aligned} L^f_{x_1,x_2}(x) &= f(x_1) + m(x - x_1) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) \\ &= f(x_1)\left(1 - \frac{x - x_1}{x_2 - x_1}\right) + f(x_2)\frac{x - x_1}{x_2 - x_1} = \frac{x_2 - x}{x_2 - x_1} f(x_1) + f(x_2)\frac{x - x_1}{x_2 - x_1}. \end{aligned}$$

Hence

$$L^f_{x_1,x_2}(x) = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2). \tag{8.11}$$

Above we recognize the numbers t_1, t_2 defined in (8.9).

Proposition 8.15. *Consider a function $f : (a, b) \to \mathbb{R}$ and $x_1, x_2 \in (a, b)$, $x_1 < x_2$. Denote by $L^f_{x_1,x_2}$ the linear function whose graph contains the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f. The following statements are equivalent.*

$$f(x) \leq L^f_{x_1,x_2}(x), \;\; \forall x \in [x_1, x_2]. \tag{8.12a}$$

$$f(x) \leq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2), \;\; \forall x \in [x_1, x_2]. \tag{8.12b}$$

$$\forall t_1, t_2 \geq 0 \;\; \textit{such that} \;\; t_1 + t_2 = 1 \;\; f(t_1x_1 + t_2x_2) \leq t_1f(x_1) + t_2f(x_2), \tag{8.12c}$$

Proof. The equivalence (8.12a) $\iff$ (8.12b) follows from (8.11). The equivalence (8.12b) $\iff$ (8.12c) follows from (8.9) and (8.10).

□

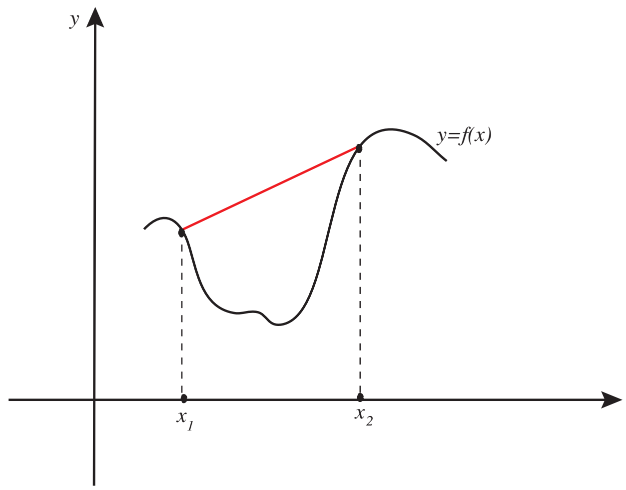

Figure 8.2. *The graph lies below the chord.*

Remark 8.16. The part of the graph of $L^f_{x_1,x_2}$ over the interval $[x_1, x_2]$ is called the *chord* of the graph of f determined by the interval $[x_1, x_2]$. The condition (8.12a) is equivalent to saying that the part of the graph of f corresponding to the interval $[x_1, x_2]$ lies below the chord of the graph determined by this interval; see Figure 8.2. □

Definition 8.17. Let $f : I \to \mathbb{R}$ be a real valued function defined on an interval I.

(i) The function f is called *convex* if, for any $x_1, x_2 \in I$, and any $t_1, t_2 \geq 0$ such that $t_1 + t_2 = 1$, we have

$$f(t_1x_1 + t_2x_2) \leq t_1f(x_1) + t_2f(x_2).$$

(ii) The function f is called *concave* if, for any $x_1, x_2 \in I$, and any $t_1, t_2 \geq 0$ such that $t_1 + t_2 = 1$, we have

$$f(t_1x_1 + t_2x_2) \geq t_1f(x_1) + t_2f(x_2).$$

□

Remark 8.18. (a) From Propositions 8.14 and 8.15 we deduce that a function $f : I \to \mathbb{R}$ is convex if and only if, for any interval $[x_1, x_2] \subset I$, the part of the graph of f determined by the interval $[x_1, x_2]$ is below the chord of the graph determined by this interval. It is concave if the graph is above the chords.

(b) Observe that if $t_1 = 1$ and $t_2 = 0$ we have $t_1x_1 + t_2x_2 = x_1$ and $t_1f(x_1) + t_2f(x_2) = f(x_1)$ and thus

$$f(t_1x_1 + t_2x_2) \leq t_1f(x_1) + t_2f(x_2).$$

is automatically satisfied. A similar thing happens when $t_1 = 0$ and $t_2 = 1$. Thus the definition of convexity is equivalent to the weaker requirement that for any $x_1, x_2 \in I$, and any *positive* t_1, t_2 such that $t_1 + t_2 = 1$, we have

$$f(t_1x_1 + t_2x_2) \leq t_1f(x_1) + t_2f(x_2).$$

(c) Observe that a function f is concave if and only if $-f$ is convex.

(d) In many calculus texts, convex functions are called *concave-up* and concave functions are called *concave-down*. □

Before we can give examples of convex functions we need to produce simple criteria for recognizing when a function is convex.

Proposition 8.14 implies that a function $f : I \to \mathbb{R}$ is convex if and only if for any $x_1, x_2 \in I$ and any $x \in (x_1, x_2)$ we have

$$f(x) \leq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

Since

$$1 = \frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1},$$

we deduce that f is convex if and only if

$$\begin{aligned}
f(x)\left(\frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1}\right) &\leq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2) \\
\iff \frac{x_2 - x}{x_2 - x_1}\big(f(x) - f(x_1)\big) &\leq \frac{x - x_1}{x_2 - x_1}\big(f(x_2) - f(x)\big) \\
\iff (x_2 - x)\big(f(x) - f(x_1)\big) &\leq (x - x_1)\big(f(x_2) - f(x)\big) \\
\iff \frac{f(x) - f(x_1)}{x - x_1} &\leq \frac{f(x_2) - f(x)}{x_2 - x}.
\end{aligned}$$

We have thus proved the following result.

Corollary 8.19. *Let $f : I \to \mathbb{R}$ be a function defined on the interval $I \subset \mathbb{R}$. The following statements are equivalent.*

(i) *The function f is convex.*

(ii) *For any $x_1, x, x_2 \in I$ such that $x_1 < x < x_2$ we have*

$$\frac{f(x) - f(x_1)}{x - x_1} \leq \frac{f(x_2) - f(x)}{x_2 - x}.$$

□

Let us observe that $\frac{f(x)-f(x_1)}{x-x_1}$ is the slope of the chord determined by $[x_1, x]$ while $\frac{f(x_2)-f(x)}{x_2-x}$ is the slope of the chord determined by $[x, x_2]$. The above result states that f is convex if and only if for any $x_1 < x < x_2$ the chord determined by $[x_1, x]$ has a smaller inclination than the chord determined by $[x, x_2]$; see Figure 8.3.

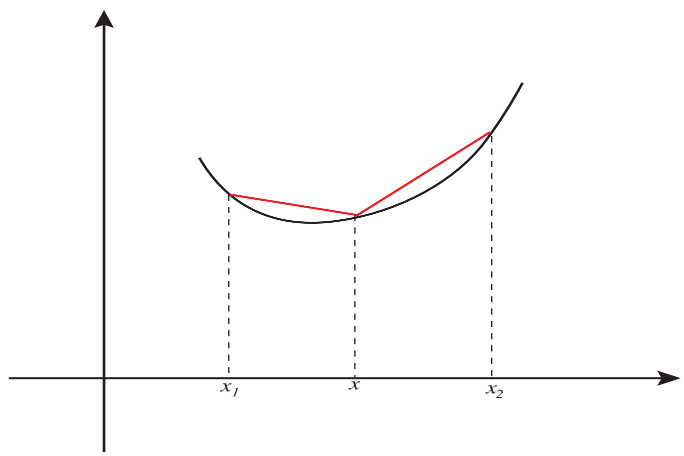

Figure 8.3. *Chords of the graph of a convex function become less inclined as they move to the right.*

Corollary 8.20. *Suppose that $f : I \to \mathbb{R}$ is a convex function. Then for any $x_1 < x_2 < x_3 < x_4 \in I$ we have*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_4) - f(x_3)}{x_4 - x_3}.$$

Proof. From Corollary 8.19 we deduce that the slope of the chord determined by $[x_1, x_2]$ is smaller than the slope of the chord determined by $[x_2, x_3]$ which in turn is smaller than the slope of the chord determined by $[x_3, x_4]$; see Figure 8.4. In other words,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2} \le \frac{f(x_4) - f(x_3)}{x_4 - x_3}.$$

□

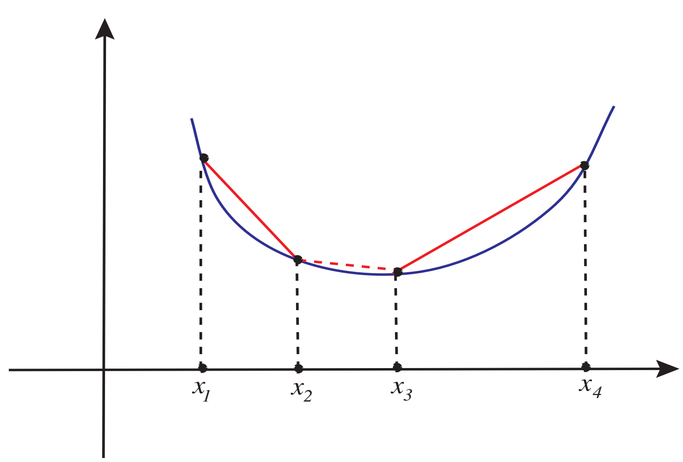

Figure 8.4. *Chords of the graph of a convex function become more inclined as they move to the right.*

Corollary 8.21. *Suppose that $f : I \to \mathbb{R}$ is a differentiable function. Then the following statements are equivalent.*

(i) *The function f is convex.*

(ii) *The derivative f' is a nondecreasing function.*

Proof. (ii) $\Rightarrow$ (i) In view of Corollary 8.19 we have to prove that for any $x_1 < x_2 < x_3 \in I$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

From Lagrange's Mean Value theorem we deduce that there exist $\xi_1 \in (x_1, x_2)$ and $\xi_2 \in (x_2, x_3)$ such that

$$f'(\xi_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \;\; f'(\xi_2) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

Since f' is nondecreasing and $\xi_1 < x_2 < \xi_2$, we deduce $f'(\xi_1) \le f'(\xi_2)$.

(i) $\Rightarrow$ (ii) We know that f is convex and we have to prove that f' is nondecreasing, i.e.,

$$x_1 < x_2 \Rightarrow f'(x_1) \le f'(x_2).$$

For $h > 0$ sufficiently small, $h < \frac{1}{2}(x_2 - x_1)$, we have

$$x_1 < x_1 + h < x_2 - h < x_2.$$

From Corollary 8.20 we deduce that slope of the chord determined by $[x_1, x_1 + h]$ is smaller than the slope of the chord determined by $[x_2 - h, x_2]$, that is,

$$\frac{f(x_1 + h) - f(x_1)}{h} \le \frac{f(x_2) - f(x_2 - h)}{h} = \frac{f(x_2 - h) - f(x_2)}{-h}.$$

Hence

$$f'(x_1) = \lim_{h \to 0+} \frac{f(x_1 + h) - f(x_1)}{h} \le \lim_{h \to 0+} \frac{f(x_2 - h) - f(x_2)}{-h} = f'(x_2).$$

□

Since a differentiable function is nondecreasing iff its derivative is nonnegative, we deduce the following useful result.

Corollary 8.22. *Suppose that $f : I \to \mathbb{R}$ is a twice differentiable function. Then the following statements are equivalent.*

(i) *The function f is convex.*

(ii) *The second derivative f'' is nonnegative, $f''(x) \ge 0$, $\forall x \in I$.*

□

Since a function is concave if and only if $-f$ is convex we deduce the following result.

Corollary 8.23. *Suppose that $f : I \to \mathbb{R}$ is a twice differentiable function. Then the following statements are equivalent.*

(i) *The function f is concave.*

(ii) *The second derivative f'' is nonpositive, $f''(x) \le 0$, $\forall x \in I$.*

□

Example 8.24. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ is convex since $f''(x) = e^x > 0$ for any $x \in \mathbb{R}$. The function $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$ is concave since

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2} < 0, \quad \forall x > 0.$$

Fix $\alpha \in \mathbb{R}$ and consider the power function

$$p : (0, \infty) \to \mathbb{R}, \quad p(x) = x^\alpha.$$

Then

$$p''(x) = \alpha(\alpha - 1)x^{\alpha-2}.$$

Note that if $\alpha(\alpha - 1) > 0$ this function is convex, if $\alpha(\alpha - 1) < 0$ this function is concave, and if $\alpha = 0$ or $\alpha = 1$ this function is both convex and concave. Thus, the function $\sqrt{x}$ is concave, while the function $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ is convex. □

8.3.1. Some classical applications of convexity. We start with a simple geometric fact.

Proposition 8.25. *Suppose that $f : I \to \mathbb{R}$ is a differentiable convex function. Then the graph of f lies above any tangent to the graph; see Figure 8.5. If additionally f' is strictly increasing, then any tangent to the graph intersects the graph at a unique point.*

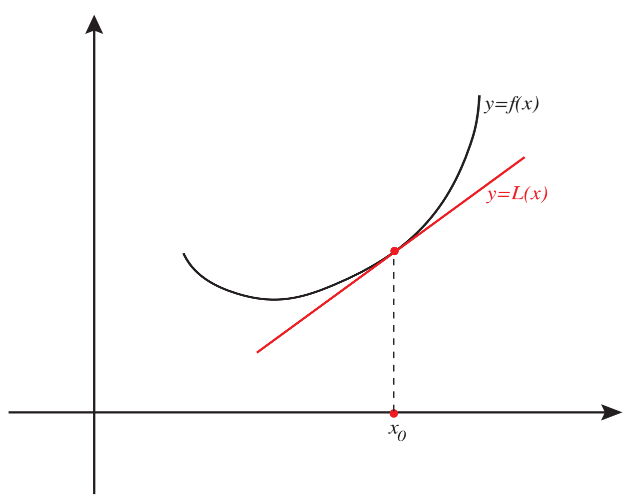

Figure 8.5. *The graph of a convex function lies above any of its tangents.*

Proof. Let $x_0 \in I$. The tangent to the graph of f at the point $(x_0, f(x_0))$ is the graph of the linearization of f at x_0 which is the function

$$L(x) = f(x_0) + f'(x_0)(x - x_0).$$

We have to prove that

$$f(x) - L(x) \geq 0, \quad \forall x \in I.$$

We have

$$f(x) - L(x) = f(x) - f(x_0) - f'(x_0)(x - x_0).$$

Suppose $x \neq x_0$. Lagrange's Mean Value Theorem implies that there exists ξ between x_0 and x such that $f(x) - f(x_0) = f'(\xi)(x - x_0)$. Hence

$$f(x) - L(x) = f'(\xi)(x - x_0) - f'(x_0)(x - x_0) = (\, f'(\xi) - f'(x_0)\,)(x - x_0).$$

We distinguish two cases.

1. $x > x_0$. Then $\xi > x_0$ and $(x - x_0) > 0$. Since f is convex, f' is increasing and thus $f'(\xi) \geq f'(x_0)$ so that

$$(\, f'(\xi) - f'(x_0)\,)(x - x_0) \geq 0.$$

Clearly if f' is strictly increasing, then $f'(\xi) > f'(x_0)$ and $(\, f'(\xi) - f'(x_0)\,)(x - x_0) > 0$.

2. $x < x_0$. Then $\xi < x_0$ and $(x - x_0) < 0$. Since f is convex, f' is increasing and thus $f'(\xi) \leq f'(x_0)$ so that

$$(\, f'(\xi) - f'(x_0)\,)(x - x_0) \geq 0.$$

Clearly if f' is strictly increasing, then $f'(\xi) < f'(x_0)$ and $(\, f'(\xi) - f'(x_0)\,)(x - x_0) > 0$ □

Example 8.26 (Newton's Method)**.** We want to describe an ingenious method devised by Newton for approximating the solutions of an equation $f(x) = 0$.

Suppose that $f : (a, b) \to \mathbb{R}$ is a C^2-function such that

$$f'(x),\;\; f''(x) > 0,\;\; \forall x \in (a, b). \tag{8.13}$$

Suppose $z_0 \in (a, b)$ satisfies

$$f(z_0) = 0.$$

The condition (8.13) implies that f is strictly increasing and thus z_0 is the unique solution of the equation $f(x) = 0$. Newton's method described one way of constructing very accurate approximations for z_0.

Here is roughly the principle behind the method. Pick an arbitrary point $x_0 \in (z_0, b)$. The linearization $L(x)$ of f at x_0 is an approximation for $f(x)$ so, intuitively, the solution of the equation $L(x) = 0$ ought to approximate the solution of the equation $f(x) = 0$. Denote by $Z(x_0)$ the solution of the equation $L(x) = 0$, i.e., the point where the tangent to the graph of f at $(x_0, f(x_0))$ intersects the horizontal axis; see Figure 8.6.

More precisely, we have $L(x) = f(x_0) + f'(x_0)(x - x_0)$ and thus,

$$L(x) = 0 \Longleftrightarrow f'(x_0)(x - x_0) = -f(x_0) \Longleftrightarrow x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$\Longleftrightarrow x = Z(x_0) = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Key Remark. *The point $Z(x_0)$ lies between z_0 and x_0, $z_0 < Z(x_0) < x_0$. In particular, $Z(x_0)$ is closer to z_0 than x_0.*

Clearly $Z(x_0) < x_0$ because $L(x_0) = f(x_0) > 0 = L(Z(x_0))$ and the linear function $L(x)$ is increasing. The assumption (8.13) implies that f is convex and f' is strictly increasing. Proposition 8.25 implies that the tangent lies below the graph, i.e.,

$$f\big(\, Z(x_0)\,\big) > L\big(\, Z(x_0)\,\big) = 0 = f(z_0).$$

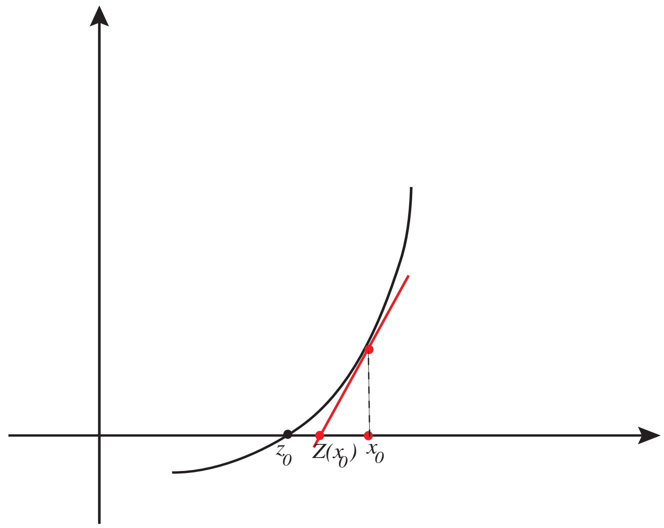

Figure 8.6. *The geometry behind Newton's method.*

Since f is strictly increasing we deduce $Z(x_0) > z_0$.

The correspondence $x_0 \mapsto Z(x_0)$ is thus a map $(z_0, b) \to (z_0, b)$ with the property that $z_0 < Z(x_0) < x_0$, $\forall x_0 \in (z_0, b)$.

We iterate this procedure. We set $x_1 = Z(x_0)$ so that $z_0 < x_1 < x_0$. Define next $x_2 = Z(x_1)$ so that $z_0 < x_2 < x_1$ and inductively

$$x_{n+1} := Z(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}, \ \ n \geq 0. \tag{8.14}$$

The above discussion shows that the sequence (x_n) is strictly decreasing and bounded below by z_0. It is therefore convergent and we set $\bar{x} = \lim x_n$. Observe that $\bar{x} \geq z_0$. Letting $n \to \infty$ in (8.14) and taking into account the continuity of f and f' we deduce

$$\bar{x} = \bar{x} - \frac{f(\bar{x})}{f'(\bar{x})} \Rightarrow \frac{f(\bar{x})}{f'(\bar{x})} = 0 \Rightarrow f(\bar{x}) = 0.$$

Since z_0 is the unique solution of the equation $f(x) = 0$ we deduce $\bar{x} = z_0$. Thus the sequence generated by Newton's iteration (8.14) converges to the unique zero of f.

Remarkably, the above sequence (x_n) converges to z_0 extremely quickly. Taylor's formula with Lagrange remainder implies that for any n there exists $\xi_n \in (z_0, x_n)$ such that

$$0 = f(z_0) = f(x_n) + f'(x_n)(z_0 - x_n) + \frac{1}{2} f''(\xi_n)(z_0 - x_n)^2.$$

Hence

$$0 = \frac{f(x_n)}{f'(x_n)} + z_0 - x_n + \frac{f''(\xi_n)}{2f'(x_n)}(z_0 - x_n)^2 \Rightarrow \underbrace{\frac{f(x_n)}{f'(x_n)} + z_0 - x_n}_{=z_0 - x_{n+1}} = -\frac{f''(\xi_n)}{2f'(x_n)}(z_0 - x_n)^2$$

Hence

$$(z_0 - x_{n+1}) = -\frac{f''(\xi_n)}{2f'(x_n)}(z_0 - x_n)^2.$$

If we denote by ε_n the error, $\varepsilon_n := x_n - z_0$ we deduce

$$\varepsilon_{n+1} = \frac{f''(\xi_n)}{2f'(x_n)} \varepsilon_n^2. \tag{8.15}$$

Thus, the error at the $(n+1)$ -th step is roughly the square of the error at the n-th step. If e.g. the error ε_n is < 0.01, then we expect $\varepsilon_{n+1} < (0.1)^2 = 0.0001$.

Let us see how this works in a simple case. Let k be a natural number ≥ 2. Consider the function

$$f : (0, \infty) \to \mathbb{R}, \;\; f(x) = x^k - 2.$$

Then

$$f'(x) = kx^{k-1}, \;\; f''(x) = k(k-1)x^{k-2}$$

so the assumption (8.13) is satisfied. The unique solution of the equation $f(x) = 0$ is the number $\sqrt[k]{2}$ and Newton's method will produce approximations for this number.

We first need to choose a number $x_0 > \sqrt[k]{2}$. How do we do this when we do not know what the number $\sqrt[k]{2}$ is?

Observe we have to choose a number x_0 such that $f(x_0) > f(\sqrt[k]{2}) = 0$, or equivalently,

$$x_0^k > 2.$$

Let's pick $x_0 = \frac{3}{2}$. Then

$$\left(\frac{3}{2}\right)^k \geq \left(\frac{3}{2}\right)^2 = \frac{9}{4} > 2.$$

Note also that $f(1) = 1^k - 2 = -1 < 0$ so that

$$1 < \sqrt[k]{2} < \frac{3}{2}$$

and thus the error

$$\varepsilon_0 = x_0 = \sqrt[k]{2} < \frac{1}{2}.$$

In this case we have

$$Z(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^k - 2}{kx^{k-1}} = \frac{k-1}{k}x + \frac{2}{kx^{k-1}}.$$

Observe that for $k = 2$ we have

$$Z(x) = \frac{x}{2} + \frac{1}{x}.$$

and the recurrence $x_{n+1} = Z(x_n)$ takes the form

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

Above, we recognize the recurrence that we have investigated earlier in Example 4.25.

For $k = 3$ the recurrence $x_{n+1} = Z(x_n)$ takes the form

$$x_{n+1} = \frac{2x_n}{3} + \frac{2}{3x_n^2}, \;\; x_0 = 1.5.$$

We have

$$x_1 = 1.296296..., \;\; x_2 = 1.260932..., \;\; x_3 = 1.25992186...,$$
$$x_4 = 1.25992104..., \;\; x_5 = 1.25992104...$$

Note that, as predicted theoretically, this sequence displays a very rapid stabilization. Thus

$$\sqrt[3]{2} \approx 1.25992.....$$

We can independently confirm the above claim by observing that

$$(1.25992)^3 = 1.999995. \qquad \square$$

Theorem 8.27 (Jensen's inequality)**.** *Suppose that $f : I \to \mathbb{R}$ is a convex function defined on an interval I. Then for any $n \in \mathbb{N}$, any $x_1, \dots, x_n \in I$ and any $t_1, \dots, t_n \geq 0$ such that*

$$t_1 + \cdots + t_n = 1$$

we have $t_1x_1 + \cdots + t_nx_n \in I$ and

$$f\big(t_1x_1 + \cdots + t_nx_n\big) \leq t_1f(x_1) + \cdots + t_nf(x_n). \tag{8.16}$$

Proof. We argue by induction on n. For $n = 1$ the inequality is trivially true, while for $n = 2$ it is the definition of convexity. We assume that the inequality is true for n and we prove it for $n + 1$.

Let $x_0, \dots, x_n \in I$ and $t_0, \dots, t_n \geq 0$ such that

$$t_0 + \cdots + t_n = 1.$$

We have to prove that

$$f(t_0x_0 + t_1x_1 + t_2x_2 + \cdots + t_nx_n) \leq t_0f(x_0) + t_1f(x_1) + t_2f(y_2) + \cdots + t_nf(y_n). \tag{8.17}$$

If one of the numbers $t_0, t_1, \dots, t_n$ is zero, then the above inequality reduces to the case n. We can therefore assume that $t_0, t_1, \dots, t_n > 0$. Consider now the real numbers

$$s_1 := t_0 + t_1, \ \ s_2 := t_2, \dots, s_n := t_n,$$

$$y_1 := \frac{t_0}{t_0 + t_1}x_0 + \frac{t_1}{t_0 + t_1}x_1, \ \ y_2 := x_2, \dots, y_n := x_n.$$

Note that

$$s_1, s_2, \dots, s_n \geq 0 \ \text{ and } \ s_1 + \cdots + s_n = 1$$

and since

$$\frac{t_0}{t_0 + t_1} + \frac{t_1}{t_0 + t_1} = 1$$

the point y_1 lies between x_0 and x_1 and thus in the interval I. From the induction assumption we deduce

$$s_1y_1 + \cdots + s_ny_n \in I,$$

and

$$f(s_1y_1 + \cdots + s_ny_n) \leq s_1f(y_1) + s_2f(y_2) + \cdots + s_nf(y_n)$$
$$= (t_0 + t_1)f\left(\frac{t_0}{t_0 + t_1}x_0 + \frac{t_1}{t_0 + t_1}x_1\right) + s_2f(y_2) + \cdots + s_nf(y_n).$$

Now observe that

$$s_1y_1 + \cdots + s_ny_n = (t_0 + t_1)\left(\frac{t_0}{t_0 + t_1}x_0 + \frac{t_1}{t_0 + t_1}x_1\right) + t_2y_2 + \cdots + t_ny_n$$
$$= t_0x_0 + t_1x_1 + t_2x_2 + \cdots + t_nx_n$$

and since f is convex

$$f\left(\frac{t_0}{t_0 + t_1}x_0 + \frac{t_1}{t_0 + t_1}x_1\right) \leq \frac{t_0}{t_0 + t_1}f(x_0) + \frac{t_1}{t_0 + t_1}f(x_1)$$

so that

$$(t_0 + t_1)\left(\frac{t_0}{t_0 + t_1}x_0 + \frac{t_1}{t_0 + t_1}x_1\right) \leq t_0f(x_0) + t_1f(x_1).$$

Putting together all of the above we deduce (8.17). $\square$

Corollary 8.28. *If $f : I \to \mathbb{R}$ is a convex function defined on an interval I, then for any $n \in \mathbb{N}$ and any $x_1, \dots, x_n \in I$ we have*

$$\boxed{f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}.} \tag{8.18}$$

Proof. Use (8.16) in which $t_1 = t_2 = \cdots = t_n = \frac{1}{n}$. □

Corollary 8.29. *Suppose that $g : I \to \mathbb{R}$ is a concave function defined on an interval I. Then for any $n \in \mathbb{N}$, any $x_1, \dots, x_n \in I$ and any $t_1, \dots, t_n \geq 0$ such that*

$$t_1 + \cdots + t_n = 1$$

we have $t_1x_1 + \cdots + t_nx_n \in I$ and

$$g\big(t_1x_1 + \cdots + t_nx_n\big) \geq t_1g(x_1) + \cdots + t_ng(x_n). \tag{8.19}$$

In particular,

$$\boxed{g\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{g(x_1) + \cdots + g(x_n)}{n}.} \tag{8.20}$$

Proof. Apply Theorem 8.27 to the convex function $f = -g$. □

Corollary 8.30 (AM-GM inequality)**.** *For any natural number n and any positive real numbers $x_1, \dots, x_n$ we have*

$$\boxed{\big(x_1 \cdots x_n\big)^{\frac{1}{n}} \leq \frac{x_1 + \cdots + x_n}{n}.}$$

The left-hand-side of the above inequality is called the geometric mean *(GM) of the numbers $x_1, \dots, x_n$, while the right-hand side is called the* arithmetic mean *(AM) of the same numbers.*

Proof. Consider the function

$$f : (0, \infty) \to \mathbb{R}, \;\; f(x) = \ln x.$$

This function is concave and (8.20) implies that

$$\ln\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{\ln x_1 + \cdots + \ln x_n}{n}.$$

Exponentiating this inequality we deduce

$$\frac{x_1 + \cdots + x_n}{n} = e^{\ln(\frac{x_1 + \cdots + x_n}{n})}$$

$$\geq e^{\frac{\ln x_1 + \cdots + \ln x_n}{n}} = e^{\frac{\ln(x_1 \cdots x_n)}{n}} = (x_1 \cdots x_n)^{\frac{1}{n}}.$$

□

Corollary 8.31 (Hölder's inequality)**.** *Fix a real number $p > 1$ and define $q > 1$ by the equality*

$$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}.$$

Then for any natural number n and any nonnegative real numbers $a_1, \dots, a_n$, $b_1, \dots, b_n$ we have

$$a_1 b_1 + \cdots + a_n b_n \leq \left(a_1^p + \cdots + a_n^p \right)^{\frac{1}{p}} \left(b_1^q + \cdots + b_n^q \right)^{\frac{1}{q}}, \tag{8.21}$$

or, using the summation notation

$$\boxed{\sum_{k=1}^n a_k b_k \leq \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n b_j^q \right)^{\frac{1}{q}}.} \tag{8.22}$$

Proof. Since $p > 1$, the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^p$, is convex. We define

$$B := b_1^q + \cdots + b_n^q,$$

$$t_k := \frac{b_k^q}{B}, \quad k = 1, \dots, n,$$

$$x_k := a_k b_k^{-\frac{1}{p-1}} B, \quad k = 1, \dots, n.$$

Observe that $t_k \geq 0$, $\forall k$ and

$$t_1 + \cdots + t_k = 1.$$

Using Jensen's inequality (8.18) we deduce that

$$\left(t_1 x_1 + \cdots + t_n x_n \right)^p \leq t_1 x_1^p + \cdots + t_n x_n^p.$$

Observe that

$$\left(t_1 x_1 + \cdots + t_n x_n \right)^p = \left(a_1 b_1^{q - \frac{1}{p-1}} + \cdots + a_n b_n^{q - \frac{1}{p-1}} \right)^p$$

$(q - \frac{1}{p-1} = 1)$

$$= \left(a_1 b_1 + \cdots + a_n b_n \right)^p.$$

Similarly

$$t_1 x_1^p + \cdots + t_n x_n^p = \frac{b_1^q}{B} a_1^p b_1^{-\frac{p}{p-1}} B^p + \cdots + \frac{b_n^q}{B} a_n^p b_n^{-\frac{p}{p-1}} B^p$$

$(q - \frac{p}{p-1} = 0)$

$$= B^{p-1} \left(a_1^p + \cdots + a_n^p \right).$$

Hence

$$\left(a_1 b_1 + \cdots + a_n b_n \right)^p \leq B^{p-1} \left(a_1^p + \cdots + a_n^p \right)$$

so that

$$a_1 b_1 + \cdots + a_n b_n \leq B^{\frac{p-1}{p}} \left(a_1^p + \cdots + a_n^p \right)^{\frac{1}{p}}$$

$$= \left(a_1^p + \cdots + a_n^p \right)^{\frac{1}{p}} \left(b_1^q + \cdots + b_n^q \right)^{\frac{1}{q}}.$$

$\square$

If in Hölder's inequality we let $p = 2$, then $q = 2$, and we obtain the following important result.

Corollary 8.32 (Cauchy-Schwarz inequality). *For any natural number n and any real numbers $x_1, \dots, x_n$, $y_1, \dots, y_n$ we have*

$$\boxed{\left|\sum_{k=1}^{n} x_k y_k\right| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} y_j^2\right)^{\frac{1}{2}}.} \tag{8.23}$$

Proof. We define

$$a_k = |x_k|, \ \ b_k = |y_k|, \ \ k = 1, \dots, n.$$

Note that $a_k^2 = x_k^2$, $b_k^2 = y_k^2$. Using Hölder's inequality with $p = q = 2$ we deduce

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} y_j^2\right)^{\frac{1}{2}}.$$

Now observe that

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \sum_{k=1}^{n} |x_k y_k|.$$

□

Corollary 8.33 (Minkowski's inequality). *For any real number $p \in [1, \infty)$, any natural number n, and any real numbers $x_1, \dots, x_n$, $y_1, \dots, y_n$ we have*

$$\boxed{\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}}.} \tag{8.24}$$

Proof. We set

$$X := \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}, \ \ Y := \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}}, \ \ Z := \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}}.$$

Clearly $X, Y, Z \geq 0$. We have to prove that $Z \leq X + Y$. This inequality is obviously true if $Z = 0$ so we assume that $Z > 0$. Note that we have

$$\begin{aligned} Z^p &= \sum_{k=1}^{n} |x_k + y_k|^p = \sum_{k=1}^{n} \underbrace{|x_k + y_k|}_{\leq |x_k| + |y_k|} \, |x_k + y_k|^{p-1} \\ &\leq \sum_{k=1}^{n} |x_k| \, |x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k| \, |x_k + y_k|^{p-1}. \end{aligned} \tag{8.25}$$

This proves (8.24) in the special case $p = 1$ so in the sequel we assume that $p > 1$. Let $q = \frac{p}{p-1}$ so that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Using Hölder's inequality we deduce that that for any $k = 1, \dots, n$ we deduce

$$\sum_{k=1}^{n} |x_k|\,|x_k + y_k|^{p-1} \leq \underbrace{\left(\sum_{k=1}^{p} |x_k|^p\right)^{\frac{1}{p}}}_{X} \underbrace{\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{p-1}{p}}}_{Z^{p-1}},$$

$$\sum_{k=1}^{n} |y_k|\,|x_k + y_k|^{p-1} \leq \underbrace{\left(\sum_{k=1}^{p} |y_k|^p\right)^{\frac{1}{p}}}_{Y} \underbrace{\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{p-1}{p}}}_{Z^{p-1}}.$$

Using the last two inequalities in (8.25) we deduce

$$Z^p \leq (X+Y)Z^{p-1} \stackrel{Z>0}{\Rightarrow} Z \leq X + Y.$$

□

Remark 8.34. Minkowski's inequality has a very useful interpretation. For a natural number n we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space whose points are called (n-dimensional) *vectors* and are defined to be n-uples

$$\boldsymbol{x} = (x_1, \dots, x_n), \;\; x_i \in \mathbb{R}, \;\; 1 \leq i \leq n.$$

The space $\mathbb{R}^n$ has a rich algebraic structure. We mention here two operations. One is the addition of vectors. Given

$$\boldsymbol{x} = (x_1, \dots, x_n), \;\; \boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$$

we define their sum $\boldsymbol{x} + \boldsymbol{y}$ to be the vector

$$\boldsymbol{x} + \boldsymbol{y} := (x_1 + y_1, \dots, x_n + y_n).$$

Another is the multiplication by a scalar. Given

$$\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \;\; t \in \mathbb{R},$$

we define

$$t\boldsymbol{x} := (tx_1, \dots, tx_n).$$

For $p \in [1, \infty)$ and $\boldsymbol{x} \in \mathbb{R}^n$ we set

$$\|\boldsymbol{x}\|_p := \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}.$$

Note that

$$\begin{aligned} &\|t\boldsymbol{x}\|_p = |t|\,\|\boldsymbol{x}\|_p, \;\; \forall t \in \mathbb{R}, \;\; \boldsymbol{x} \in \mathbb{R}^n, \\ &\|\boldsymbol{x}\|_p \geq 0, \;\; \forall \boldsymbol{x} \in \mathbb{R}^n, \\ &\|\boldsymbol{x}\|_p = 0 \Longleftrightarrow \boldsymbol{x} = (0, 0, \dots, 0). \end{aligned} \tag{8.26}$$

Minkowski's inequality is then equivalent to the *triangle inequality*

$$\|\boldsymbol{x} + \boldsymbol{y}\|_p \leq \|\boldsymbol{x}\|_p + \|\boldsymbol{y}\|_p, \;\; \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n. \tag{8.27}$$

A function $\mathbb{R}^n \to \mathbb{R}$ that associates to a vector $\mathbb{R}$ a real number $\|\boldsymbol{x}\|$ satisfying (8.26) and (8.27) is called a *norm* on $\mathbb{R}^n$. Minkowski's inequality can be interpreted as saying that for any $p \in [1, \infty)$ the correspondence

$$\mathbb{R}^n \ni \boldsymbol{x} \mapsto \|\boldsymbol{x}\|_{\boldsymbol{p}} \in [0, \infty),$$

defines a norm on $\mathbb{R}^n$.

Note that (8.27) implies that for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ we have

$$\|\boldsymbol{u}-\boldsymbol{w}\|_p \leq \|\boldsymbol{u}-\boldsymbol{v}\|_p + \|\boldsymbol{v}-\boldsymbol{w}\|_p, \tag{8.28}$$

since

$$\underbrace{(\boldsymbol{u}-\boldsymbol{v})}_{\boldsymbol{x}} + \underbrace{(\boldsymbol{v}-\boldsymbol{w})}_{\boldsymbol{y}} = \underbrace{(\boldsymbol{u}-\boldsymbol{w})}_{\boldsymbol{x}+\boldsymbol{y}}. \qquad \square$$

8.4. How to sketch the graph of a function

Differential calculus can be quite useful in producing sketches of the graphs of functions. Instead of giving a detailed description of the steps that need to be taken to produce a sketch of a graph, we will outline a few general principles and illustrate them on a few examples.

In sketching the graph of a function $f(x)$, one needs to look at certain distinguishing features.

- Locate the intersections of f with the coordinate axes, if possible.
- Locate, if possible, the critical points of f, i.e., the points x such that $f'(x) = 0$.
- Locate the intervals where f is increasing and the intervals where f is decreasing, if possible.
- Locate the intervals where f is convex, and the intervals where f is concave, if possible. The endpoints of such intervals are found among the solutions of the equation.

  $$f''(x) = 0.$$

  Sometimes solving this equation explicitly may not be possible.
- Locate the asymptotes, if any.

Example 8.35 (Cubic polynomials)**.** Consider an arbitrary cubic polynomial

$$p : \mathbb{R} \to \mathbb{R}, \;\; p(x) = x^3 + a_2x^2 + a_1x + a_0,$$

where a_0, a_1, a_2 are given real numbers. We would like to describe the general appearance of the graph of p and analyze how it depends on the coefficients $a_0, a1, a_2$. Observe first that

$$\lim_{x\to\pm\infty} p(x) = \pm\infty.$$

The graph intersects the y-axis at $y = a_0$. The intersection with the x-axis is difficult to find because the equation $p(x) = 0$ is difficult to solve. Instead, we will try to find the critical points of $p(x)$ i.e., the solutions of the equation $p'(x) = 0$.

$$3x^2 + 2a_2x + a_1 = 0. \tag{8.29}$$

The function $p'(x)$ has a global minimum achieved at the point μ defined by the equation

$$p''(\mu) = 0 \Longleftrightarrow 6\mu + 2a_2 = 0 \Longleftrightarrow \mu = -\frac{a_2}{3}.$$

The function $p'(x)$ is decreasing on the interval $(-\infty, \mu]$ and increasing on $[\mu, \infty)$. Thus $p(x)$ is concave on $(-\infty, \mu]$ and convex on $[\mu, \infty)$. The point μ is an inflection point of p.

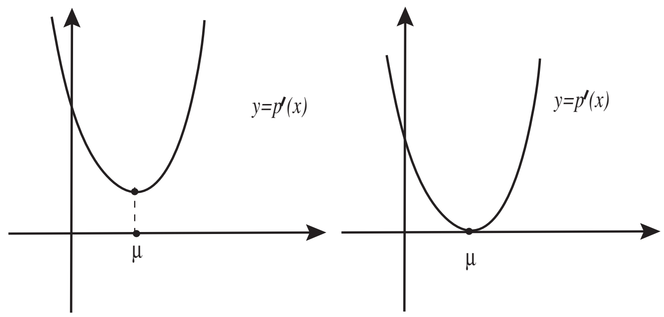

Figure 8.7. $\Delta = 4a_2^2 - 12a_1 \leq 0$.

The general theory of quadratic equations tells us that (8.29) can have zero, one or two solutions depending on whether $\Delta = 4a_2^2 - 12a_1$ is negative, zero or positive. These situations are depicted in Figure 8.7 and 8.8.

If p has no critical points, as in the left-hand side of Figure 8.7, then $p'(x) > 0$ for any $x \in \mathbb{R}$. This shows that p is increasing. Similarly, if p has a single critical point, then again $p(x)$ is increasing. In both cases, the graph of p looks like the left-hand side of Figure 8.9.

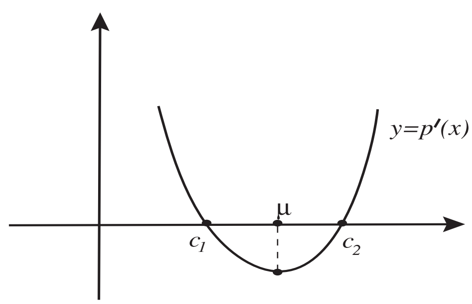

Figure 8.8. $\Delta = 4a_2^2 - 12a_1 > 0$.

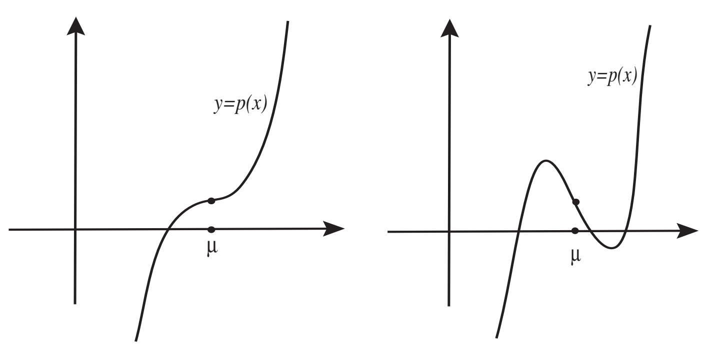

Figure 8.9. *The graph of* $y = x^3 + a_2x^2 + a_1x + a_0$.

If $p(x)$ has two critical points $c_1 < c_2$, then $p'(x) < 0$ on (c_1, c_2) and positive on $(-\infty, c_1) \cup (c_2, \infty)$; see Figure 8.8. The point c_1 is a local max of p and c_2 is a local min of

p. The inflection point μ is the midpoint of the interval $[c_1, c_2]$. The graph of p is depicted on the right-hand side of Figure 8.9. □

Example 8.36. Consider the function

$$f(x) = \frac{x^2+1}{x^2-3x+2}.$$

We have not specified its domain so it is understood to consist of all the x for which the fraction

$$\frac{x^2+1}{x^2-3x+2}$$

is well defined. The only problems are the points where the denominator vanishes,

$$x^2-3x+2 = 0 \iff x = 1 \quad \vee \quad x = 2.$$

Thus the domain is

$$(-\infty, 1) \cup (1,2) \cup (2, \infty).$$

The points 1 and 2 are also points where the vertical asymptotes could be located. We will investigate this issue later.

We have

$$f'(x) = \frac{(2x)(x^2-3x+2)-(x^2+1)(2x-3)}{(x^2-3x+2)^2} = \frac{2x^3-6x^2+4x-(2x^3-3x^2+2x-3)}{(x^2-3x+2)^2}$$
$$= \frac{-3x^2+2x+3}{(x^2-3x+2)^2}.$$

The derivative vanishes when $3x^2-2x-3=0$. The roots of this quadratic polynomial are

$$\frac{2\pm\sqrt{4+36}}{6} = \frac{2\pm\sqrt{40}}{6} = \frac{2\pm 2\sqrt{10}}{6} = \frac{1\pm\sqrt{10}}{3}.$$

One root is obviously negative. Since $3 < \sqrt{10} < 4$ we deduce

$$1 < \frac{1+\sqrt{10}}{3} < \frac{5}{3} < 2.$$

The intersection with the y-axis is obtained by computing $f(0) = \frac{1}{2}$. There is no intersection with the x axis since the numerator does not vanish. We have already detected several remarkable points

$$-\infty,\ c_1 = \frac{1-\sqrt{10}}{3},\ 1,\ c_2 = \frac{1+\sqrt{10}}{3},\ 2,\ \infty.$$

Observe that

$$\lim_{x\to\pm\infty} \frac{x^2+1}{x^2-3x+2},$$

so the horizontal line $y=1$ is a horizontal asymptote for $f(x)$ at $\pm\infty$. We do not investigate the second derivative because it requires a substantial amount of work, with little payoff.

Table 8.1 organizes the information we have collected. The exclamation signs indicate that the corresponding functions are not defined at those points. As x approaches 1 from the left, the function $f(x)$ is increasing and

$$\lim_{x\to 1-} f(x) = \infty.$$

x	$-\infty$		c_1		1		c_2		2		∞
(x^2-3x+2)	∞	$++$	$+$	$++$	0	$---$	$-$	$--$	0	$++$	∞
$-3x^2+2x+3$	$-\infty$	$--$	0	$++$	$+$	$++$	0	$--$	$-$	$--$	$-\infty$
$f'(x)$	$-$	$--$	0	$++$	!	$++$	0	$--$	!	$--$	$-$
$f(x)$	1	$\searrow$	min	$\nearrow$	!	$\nearrow$	max	$\searrow$	!	$\searrow$	1

Table 8.1. *Organizing all the relevant data.*

Similarly, the table shows

$$\lim_{x\to 1+} f(x) = -\infty, \quad \lim_{x\to 2-} f(x) = -\infty, \quad \lim_{x\to 2+} f(x) = \infty.$$

This shows that the vertical lines $x = 1$ and $x = 2$ are asymptotes of $f(x)$. Figure 8.10 contains a sketch of the graph of the function $f(x)$.

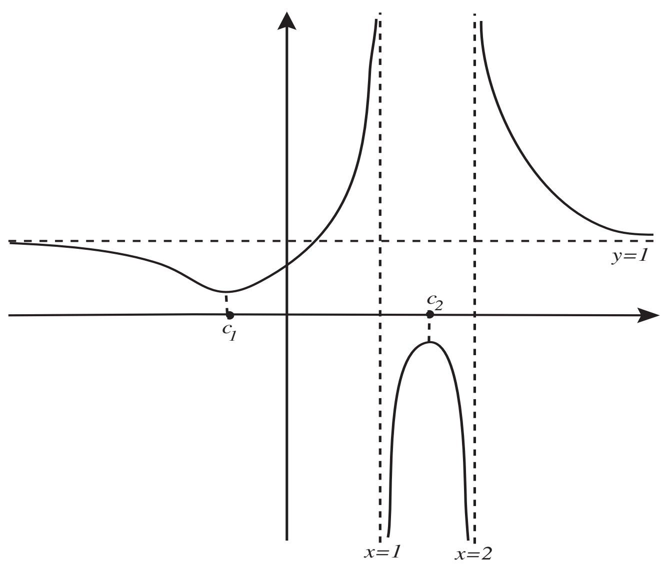

Figure 8.10. *The graph of* $\frac{x^2+1}{x^2-3x+2}$.

□

Some functions admit *inclined asymptotes.*

Definition 8.37. (a) The line $y = mx + b$ is the asymptote of $f(x)$ at ∞ if

$$\lim_{x\to\infty} \frac{f(x)}{x} = m \quad \text{and} \quad \lim_{x\to\infty} (f(x) - mx) = b.$$

(b) The line $y = mx + b$ is the asymptote of $f(x)$ at $-\infty$ if

$$\lim_{x\to-\infty} \frac{f(x)}{x} = m \quad \text{and} \quad \lim_{x\to-\infty} (f(x) - mx) = b.$$ □

Example 8.38. The function

$$f(x) = \frac{x^5 + 2x^4 + 3x^3 + 4x + 5}{x^4 + 1}$$

admits an inclined asymptote $y = mx + b$ as $x \to \infty$. The slope m can be found from the equality

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = 1,$$

and b can be found from the equality

$$b = \lim_{x\to\infty} \big(f(x) - x \big) = \lim_{x\to\infty} \frac{x^5 + 2x^4 + 3x^3 + 4x + 5 - x(x^4+1)}{x^4+1}$$

$$= \lim_{x\to\infty} \frac{2x^4 + 3x^3 + 3x + 5}{x^4+1} = 2. \qquad \square$$

8.5. Antiderivatives

Definition 8.39. Suppose that $f : I \to \mathbb{R}$ is a function defined on an interval $I \subset \mathbb{R}$. A function $F : I \to \mathbb{R}$ is called an *antiderivative* or *primitive* of f on I if F is differentiable, and

$$F'(x) = f(x), \ \ \forall x \in I. \qquad \square$$

Example 8.40. (a) The function x^2 is an antiderivative of $2x$ on $\mathbb{R}$. Similarly, the function $\sin x$ is an antiderivative of $\cos x$ on $\mathbb{R}$. $\square$

Observe that if $F(x)$ is an antiderivative of a function $f(x)$ on an interval I, then for any constant $C \in \mathbb{R}$ the function $F(x) + C$ is also an antiderivative of $f(x)$ on I. The converse is also true.

Proposition 8.41. *If F_1, F_2 are antiderivatives of the function $f : I \to \mathbb{R}$, then $F_1 - F_2$ is constant.*

Proof. Observe that $(F_1 - F_2)' = F_1' - F_2' = f - f = 0$ and Corollary 7.30 implies that $F_1 - F_2$ is constant on I. $\square$

Definition 8.42. Given a function $f : I \to \mathbb{R}$ we denote by $\int f(x)dx$ the *collection* of all the antiderivatives of f on I. Usually $\int f(x)dx$ is referred to as the *indefinite integral* of f. $\square$

For example,

$$\int \cos x \, dx = \sin x + C, \ \ \int 2x \, dx = x^2 + C.$$

Table 8.2 describes the antiderivatives of some basic functions.

Note that if $f :\to \mathbb{R}$ is a differentiable function, then f is an antiderivative of f' so that

$$\int f'(x)dx = f(x) + c. \tag{8.30}$$

Observing that $f'(x)dx = df$ we rewrite the above equality as

$$\int df = f + C. \tag{8.31}$$

$f(x)$	$\int f(x)dx$
x^n, $(x \in \mathbb{R},\, n \in \mathbb{Z},\ n \geq 0)$	$\frac{x^{n+1}}{(n+1)} + C$
$\frac{1}{x^n}$ $(x \neq 0,\, n \in \mathbb{N},\ \ n > 1)$	$-\frac{1}{(n-1)x^{n-1}} + C$
x^α, $(\alpha \in \mathbb{R},\, \alpha \neq -1,\, x > 0)$	$\frac{x^{\alpha+1}}{\alpha+1} + C$
$1/x$, $x \neq 0$	$\ln\lvert x\rvert + C$
e^x, $(x \in \mathbb{R})$	$e^x + C$
$\sin x$, $(x \in \mathbb{R})$	$-\cos x + C$
$\cos x$, $(x \in \mathbb{R})$	$\sin x + C$
$1/\cos^2 x$	$\tan x + C$
$\frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$	$\arcsin x + C$
$\frac{1}{1+x^2}$, $x \in \mathbb{R}$	$\arctan x + C$
$\int \frac{1}{\sqrt{x^2 \pm 1}}dx$, $x^2 \pm 1 > 0$	$\ln\left\lvert x + \sqrt{x^2 \pm 1}\right\rvert + C$

Table 8.2. Table of integrals.

In general, the computation of an antiderivative is a more challenging task that cannot always be completed. There are a few tricks and a few classes of functions for which this task is feasible. We will spend the remainder of this section discussing a few frequently encountered techniques for computing antiderivatives.

Proposition 8.43 (Linearity)**.** *Suppose $f, g : I \to \mathbb{R}$ and $a, b \in \mathbb{R}$. If $F, G : I \to \mathbb{R}$ are antiderivatives of f and respectively g on I, then $aF + bG$ is an antiderivative of $af + bg$ on I. We write this in condensed form*

$$\int (af + bg)dx = a \int f dx + b \int g dx.$$

Proof.

$$(aF + bG)' = aF' + bG' = af + bg.$$

□

Example 8.44.

$$\int (3 + 5x + 7x^2)dx = 3 \int dx + 5 \int x dx + 7 \int x^2 dx = 3x + \frac{5}{2}x^2 + \frac{7}{3}x^3 + C.$$ □

Proposition 8.45 (Integration by parts)**.** *Suppose that $f, g : I \to \mathbb{R}$ are two differentiable functions. If the function $f(x)g'(x)$ admits antiderivatives on I, then so does the function $f'(x)g(x)$ and moreover*

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx. \tag{8.32}$$

Proof. The function $(fg)' = f'g + fg'$ admits antiderivatives and thus the difference

$$(fg)' - fg' = f'g$$

admits antiderivatives. Moreover,

$$fg = \int (fg)'dx = \int (f'g + fg')dx = \int f'gdx + \int fg'dx \Rightarrow \int fg'dx = fg - \int gf'dx.$$

□

Let us observe that we can rewrite (8.32) in the simpler form

$$\int fdg = fg - \int gdf. \tag{8.33}$$

Example 8.46. (a) We can use integration by parts to find the antiderivatives of $\ln x$, $x > 0$. We have

$$\int \ln xdx = (\ln x)x - \int xd(\ln x) = x\ln x - \int x\frac{dx}{x}$$
$$= x\ln x - \int dx = x\ln x - x + C.$$

(b) For $a \in \mathbb{R}$ consider the indefinite integrals

$$I_a = \int e^{ax}\cos xdx, \quad J_a = \int e^{ax}\sin xdx.$$

We have

$$I_a = \int e^{ax}d(\sin x) = e^{ax}\sin x - \int \sin xd(e^{ax}) = e^{ax}\sin x - \int ae^{ax}\sin xdx$$
$$= e^{ax}\sin x - aJ_a.$$

Similarly we have

$$J_a = \int e^{ax}d(-\cos x) = -e^{ax}\cos x + \int \cos xd(e^{ax}) = -e^{ax}\cos x + a\int e^{ax}\cos xdx$$
$$= -e^{ax}\cos x + aI_a.$$

We deduce

$$I_a = e^{ax}\sin x - a(-e^{ax}\cos x + aI_a) = e^{ax}\sin x + ae^{ax}\cos x - a^2 I_a,$$

so that

$$(a^2+1)I_a = e^{ax}\sin x + ae^{ax}\cos x,$$

which shows that

$$I_a = \frac{1}{a^2+1}\big(\, e^{ax}\sin x + ae^{ax}\cos x\,\big) + C. \tag{8.34}$$

From this we deduce

$$J_a = aI_a - e^{ax}\cos x = \frac{a}{a^2+1}\big(\, e^{ax}\sin x + ae^{ax}\cos x\,\big) - e^{ax}\cos x + C,$$

so that

$$J_a = \frac{1}{a^2+1}\Big(ae^{ax}\sin x - e^{ax}\cos x \Big) + C. \tag{8.35}$$

(c) For any nonnegative integer n we consider the indefinite integral

$$I_n = \int x^n e^x dx.$$

Note that

$$I_0 = \int e^x dx = e^x + c.$$

In general we have

$$I_{n+1} = \int x^{n+1} d(e^x) = x^{n+1}e^x - \int e^x d(x^{n+1}) = x^{n+1}e^x - (n+1)\int x^n e^x dx$$

so that

$$I_{n+1} = x^{n+1}e^x - (n+1)I_n, \;\; \forall n = 0,1,2,\dots.. \tag{8.36}$$

If we let $n = 0$ in the above equality we deduce

$$I_1 = xe^x - I_0 = xe^x - e^x + C, \tag{8.37}$$

Using $n = 1$ in (8.36) we obtain

$$I_2 = x^2e^x - 2I_1 = x^2e^x - 2xe^x + 2e^x + C.$$

This suggests that in general $I_n = P_n(x)e^x + C$, where $P_n(x)$ is a polynomial of degree n. For example,

$$P_0(x) = 1, \;\; P_1(x) = (x-1), \;\; P_2(x) = x^2 - 2x + 2.$$

The equality (8.36) shows that

$$P_{n+1}(x) = x^{n+1} - (n+1)P_n(x), \;\; \forall n = 0,1,2,\dots. \tag{8.38}$$

(d) Let us now explain how to compute the integrals

$$A_n = \int \frac{dx}{(x^2+1)^n}.$$

Note that

$$A_1 = \int \frac{dx}{x^2+1} = \arctan x + C.$$

In general

$$\begin{aligned} A_n &= \int (x^2+1)^{-n} dx = x(x^2+1)^{-n} - \int x d\Big((x^2+1)^{-n} \Big) \\ &= \frac{x}{(x^2+1)^n} - \int x \frac{-2nx}{(x^2+1)^{n+1}} dx = \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2}{(x^2+1)^{n+1}} dx \\ &= \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2+1-1}{(x^2+1)^{n+1}} dx \\ &= \frac{x}{(x^2+1)^n} + 2n\int \frac{1}{(x^2+1)^n} dx - 2n\int \frac{1}{(x^2+1)^{n+1}} dx \\ &= \frac{x}{(x^2+1)^n} + 2nA_n - 2nA_{n+1}. \end{aligned}$$

Hence

$$A_n = \frac{x}{(x^2+1)^n} + 2nA_n - 2nA_{n+1},$$

so that

$$2nA_{n+1} = \frac{x}{(x^2+1)^n} + (2n-1)A_n,$$

and thus

$$\boxed{A_{n+1} = \frac{1}{2n}\frac{x}{(x^2+1)^n} + \frac{(2n-1)}{2n}A_n}. \tag{8.39}$$

For example,

$$\int \frac{1}{(x^2+1)^2}dx = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan x + C. \qquad \square$$

Proposition 8.47 (Integration by substitution)**.** *Suppose that $u : I \to J$ and $f : J \to \mathbb{R}$ are differentiable functions. Then the function $f'(u(x))u'(x)$ admits antiderivatives on I and*

$$\int f'(u(x))u'(x)dx = \int f'(u)du = \int df = f(u) + C, \quad u = u(x). \tag{8.40}$$

Proof. The chain formula shows that $f'(u(x))u'(x)$ is the derivative of $f(u(x))$ so that $f(u(x))$ is an antiderivative of $f'(u(x))u'(x)$. $\square$

Example 8.48. (a) To find an antiderivative of xe^{x^2} we use the change in variables $u = x^2$. Then

$$du = 2xdx \Rightarrow xdx = \frac{du}{2}$$

so that

$$\int e^{x^2}xdx = \int e^u\frac{du}{2} = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

(b) Let us compute an antiderivative of $\tan x = \frac{\sin x}{\cos x}$ on an interval I where $\cos x \neq 0$. We distinguish two cases.

1. $\cos x > 0$ on I. We make the change in variables $u = \cos x$ so that $u > 0$, and $du = -\sin xdx$. We have

$$\int \frac{\sin x}{\cos x}dx = -\int \frac{du}{u} = -\ln u + C = -\ln\cos x + C = -\ln|\cos x| + C.$$

2. $\cos x < 0$ on I. We make the change in variables $v = -\cos x$ so that $v > 0$ and $dv = \sin xdx$. We have

$$\int \frac{\sin x}{\cos x}dx = \int \frac{dv}{-v} = -\ln v + C = -\ln(-\cos x) + C = -\ln|\cos x| + C.$$

Thus, in either case we have

$$\int \tan xdx = -\ln|\cos x| + C. \tag{8.41}$$

(c) To compute the integral

$$\int (ax+b)^n dx, \quad n \in \mathbb{N}, \quad a > 0,$$

we make the change in variables $u = ax + b$. Then $du = adx$ so that $dx = \frac{1}{a}du$ and we have

$$\int (ax+b)^n dx = \frac{1}{a}\int u^n du = \frac{1}{a(n+1)}u^{n+1} + C = \frac{1}{a(n+1)}(ax+b)^{n+1} + C.$$

(d) To compute the integral

$$\int \frac{1}{(ax+b)^n}dx, \quad a \neq 0, n \in \mathbb{N}$$

we again make the change in variables $u = (ax+b)$ and we deduce

$$\int \frac{1}{(ax+b)^n}dx = \frac{1}{a}\int \frac{1}{u^n}du = C + \begin{cases} \frac{1}{a}\ln|u|, & n = 1 \\ \\ \frac{1}{a(1-n)u^{n-1}}, & n > 1. \end{cases}, \quad u = ax+b.$$

(e) To compute the integral

$$B_n := \int \frac{x}{(x^2+1)^n}dx$$

We make the change in variables $u = x^2+1$. Then $du = 2xdx$ so that $xdx = \frac{1}{2}du$ and thus

$$\int \frac{x}{(x^2+1)^n}dx = \frac{1}{2}\int \frac{1}{u^n}du = C + \frac{1}{2} \times \begin{cases} \ln u, & n = 1 \\ \\ \frac{1}{(1-n)u^{n-1}}, & n > 1. \end{cases}, \quad u = x^2+1.$$

(f) The integrals of the form

$$\int (\sin x)^m (\cos x)^{2k+1}dx, \quad k, m \in \mathbb{Z}_{\geq 0},$$

are found using the change in variables $u = \sin x$. Then

$$du = \cos x dx, \quad (\cos x)^{2k+1}dx = (\cos^2 x)^k \cos x dx = (1-\sin^2 x)^k d(\sin x) = (1-u^2)^k du$$

and

$$\int (\sin x)^m (\cos x)^{2k+1}dx = \int u^m (1-u^2)^k du.$$

Similarly, the integrals of the form

$$\int (\cos x)^m (\sin x)^{2k+1}dx, \quad m, k \in \mathbb{Z}_{\geq 0},$$

are found using the change in variables $v = \cos x$. Then

$$\int (\cos x)^m (\sin x)^{2k+1}dx = -\int v^m (1-v^2)^k dv.$$

(g) The integrals of the form

$$\int (\sin x)^{2m} (\cos x)^{2k}dx, \quad k, m \in \mathbb{Z}_{\geq 0}$$

are a bit trickier to compute. There are two possible strategies.

One strategy is based on the trigonometric identities

$$\boxed{\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}}.$$

Using the change in variables $u = 2x$, so that

$$du = 2dx \Rightarrow dx = \frac{1}{2}du$$

we deduce

$$\int (\sin x)^{2m}(\cos x)^{2k}dx = \frac{1}{2^{m+k+1}}\int (1-\cos u)^m(1+\cos u)^k du.$$

The last integral involves <u>*smaller*</u> powers in $\cos u$. For example

$$\int \cos^4 x dx = \int \left(\frac{1+\cos u}{2}\right)^2 \frac{du}{2}, \quad u = 2x,$$

$$= \frac{1}{8}\int (1+2\cos u+\cos^2 u)du = \frac{1}{8}u + \frac{1}{4}\sin u + \frac{1}{8}\int \cos^2 u du$$

$(v = 2u = 4x)$

$$= \frac{1}{8}u + \frac{1}{4}\sin u + \frac{1}{8}\int \left(\frac{1+\cos v}{2}\right)\frac{dv}{2} = \frac{1}{8}u + \frac{1}{4}\sin u + \frac{1}{32}\int (1+\cos v)dv$$

$$= \frac{1}{4}x + \frac{1}{4}\sin(2x) + \frac{v}{32} + \frac{1}{32}\sin v + C$$

$$= \frac{x}{4} + \frac{1}{4}\sin(2x) + \frac{x}{8} + \frac{1}{32}\sin(4x) + C = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.$$

One other possible strategy is to use the change in variables $u = \tan x$. Then

$$\cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+u^2}, \quad \sin^2 x = \cos^2 x\tan^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{u^2}{1+u^2}$$

$$du = d(\tan x) = (1+\tan^2 x)dx = (1+u^2)dx \Rightarrow dx = \frac{du}{1+u^2}.$$

We deduce

$$\int (\sin x)^{2m}(\cos x)^{2k}dx = \int \left(\frac{u^2}{1+u^2}\right)^m \left(\frac{1}{1+u^2}\right)^k \frac{du}{1+u^2}$$

$$= \int \frac{u^{2m}}{(1+u^2)^{m+k+1}}du.$$

Thus we need to know how to compute integrals of the form

$$J(m,n) = \int \frac{u^{2m}}{(1+u^2)^n}du, \quad 0 \leq m < n, \quad m,n \in \mathbb{Z}.$$

Observe first that when $m = 0$ the integrals $J(0,n)$ coincide with the integrals A_n of (8.39). The general case can be gradually reduced to the case $J(0,n)$ by observing that

$$J(m,n) = \int \frac{u^{2m}+u^{2m-2}-u^{2m-2}}{(1+u^2)^n}du = \int \frac{u^{2m-2}(1+u^2)}{(1+u^2)^n} - \int \frac{u^{2m-2}}{(1+u^2)^n}$$

$$= \int \frac{u^{2m-2}}{(1+u^2)^{n-1}} - J(m-1,n)$$

so that

$$\boxed{J(m,n) = J(m-1,n-1) - J(m-1,n).} \qquad \square$$

The examples discussed above will allow us to describe a procedure for computing the antiderivatives of any *rational function*, i.e., a function $f(x)$ of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials. Theoretically, the procedure works for any rational function, but the practical implementation can lead to complex computations. Such computation is possible because any rational function can be written as a sum of rational functions of the following simpler types.

Type I.

$$ax^n, \;\; a \in \mathbb{R}, \;\; n = 0, 1, 2, \dots .$$

Type II.

$$\frac{a}{(x-r)^n}, \;\; c, r \in \mathbb{R}, \;\; n \in \mathbb{N}.$$

Type III.

$$\frac{bx+c}{\big(\,(x-r)^2 + a^2\,\big)^n}, \;\; a, b, c, r \in \mathbb{R}, \;\; n \in \mathbb{N}.$$

If the degree of the numerator $P(x)$ is smaller than the degree of the denominator $Q(x)$, then only the Type II and Type III functions appear in the decomposition of $\frac{P(x)}{Q(x)}$. The functions of Type II and III are also known as *partial fractions* or *simple fractions*.

Actually finding the decomposition of a rational function as a sum of simple fractions requires a substantial amount of work and it is not very practical for more complicated rational functions. For this reason we will not discuss this technique in great detail.

The primitives of a function of Type I are known. More precisely

$$\int ax^n dx = \frac{a}{n+1} x^{n+1} + C.$$

The primitives of the functions of Type II where computed in Example 8.48(e). To deal with the Type III functions we make a change in variables

$$x - r = at \Longleftrightarrow x = at + r.$$

Then

$$dx = adt, \;\; bx + c = b(at+r) + \beta = abt + rb + c,$$
$$(x-r)^2 + a^2 = a^2t^2 + a^2 = a^2(t^2+1),$$

so that

$$\int \frac{bx+c}{\big(\,(x-r)^2+a^2\,\big)^n} dx = \int \frac{abt+rb+c}{a^{2n}(t^2+1)^n} adt$$
$$= \frac{b}{a^{2n-2}} \int \frac{t}{(t^2+1)^n} dt + \frac{rb+c}{a^{2n-1}} \int \frac{1}{(t^2+1)^n} dt.$$

The computation of integral

$$\int \frac{1}{(t^2+1)^n} dt$$

is described in (8.39), while the computation of the integral

$$\int \frac{t}{(t^2+1)^n} dt$$

as described in Example 8.48(e).

Let us illustrate this strategy on a simple example.

Example 8.49. Consider the rational function

$$f(x) = \frac{1}{(x-1)^2(x^2+2x+2)}.$$

Let us observe that

$$x^2+2x+2 = (x+1)^2+1^2.$$

The function admits a decomposition of the form

$$\frac{1}{(x-1)^2(x^2+2x+2)} = f(x) = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{B_1x+C_1}{x^2+2x+2}.$$

Multiplying both sides by $(x-1)^2(x^2+2x+2)$ we deduce that for any $x \in \mathbb{R}$ we have

$$1 = A_1(x-1)(x^2+2x+2) + A_2(x^2+2x+2) + (B_1x+C_1)(x-1)^2.$$

$$= A_1(x^3+2x^2+2x-x^2-2x-2) + A_2(x^2+2x+2) + (B_1x+C_1)(x^2-2x+1)$$

$$= A_1(x^3+x^2-2) + A_2(x^2+2x+2) + (B_1x^3-2B_1x^2+B_1x+C_1x^2-2C_1x+C_1)$$

$$= (A_1+B_1)x^3 + (A_1+A_2-2B_1+C_1)x^2 + (2A_2+B_1-2C_1)x - 2A_1 + 2A_2 + C_1.$$

This implies

$$\begin{cases} A_1+B_1 &= 0 \\ A_1+A_2-2B_1+C_1 &= 0 \\ 2A_2+B_1-2C_1 &= 0 \\ -2A_1+2A_2+C_1 &= 1. \end{cases}$$

From the first equality we deduce $A_1 = -B_1$ and using this in the last three equalities above we deduce

$$\begin{cases} A_2-3B_1+C_1 &= 0 \\ 2A_2+B_1-2C_1 &= 0 \\ 2A_2+2B_1+C_1 &= 1. \end{cases}$$

From the first equality we deduce $A_2 = 3B_1 - C_1$. Using this in the last two equalities we deduce

$$\begin{cases} 7B_1-4C_1 &= 0 \\ 8B_1-C_1 &= 1. \end{cases}$$

Hence,

$$\frac{7}{4}B_1 = C_1 = 8B_1 - 1 \Rightarrow \frac{25}{4}B_1 = 1 \Rightarrow B_1 = \frac{4}{25},\ \ C_1 = \frac{7}{25} \Rightarrow A_1 = -\frac{4}{25},$$

$$A_2 = 3B_1 - C_1 = \frac{12}{25} - \frac{7}{25} = \frac{1}{5}.$$

Hence

$$\frac{1}{(x-1)^2(x^2+2x+2)} = -\frac{4}{25(x-1)} + \frac{1}{5(x-1)^2} + \frac{4x+7}{25\big((x+1)^2+1^2\big)}. \qquad \square$$

Example 8.50 (First order linear differential equations)**.** A quantity u that depends on time can be viewed as a function

$$u : I \to \mathbb{R}, \ \ t \mapsto u(t),$$

where $I \subset \mathbb{R}$ is a time interval. We say that u satisfies a *linear first order differential equation* if u is differentiable and it satisfies an equality of the form

$$u'(t) + r(t)u(t) = f(t), \ \ t \in I, \tag{8.42}$$

where $r, f : I \to \mathbb{R}$ are some given functions. Solving a differential equation such as (8.42) means finding all the differentiable functions $u : I \to \mathbb{R}$ satisfying the above equality. Let us look at some special examples.

(a) If $r(t) = 0$ for any $t \in I$, then (8.42) has the simpler form $u'(t) = f(t)$, so that $u(t)$ must be an antiderivative of $f(t)$.

(b) The general case. Suppose that $r(t)$ admits antiderivatives on I. The differential equation (8.42) is solved as follows.

Step 1. Choose one antiderivative $R(t)$ of $r(t)$, i.e., a function $R(t)$ such that $R'(t) = r(t)$.

Step 2. Multiply both sides of (8.42) by $e^{R(t)}$. We obtain the equality

$$e^{R(t)}u'(t) + e^{R(t)}r(t)u(t) = f(t)e^{R(t)}.$$

Now observe that the left-hand side of the above equality is the derivative of $e^{R(t)}u(t)$,

$$\big(e^{R(t)}u(t) \big)' = e^{R(t)}u'(t) + e^{R(t)}R'(t)u(t) = e^{R(t)}u'(t) + e^{R(t)}r(t)u(t) = f(t)e^{R(t)}.$$

This shows that $e^{R(t)}u(t)$ is an antiderivative of $f(t)e^{R(t)}$.

Step 3. Find one antiderivative $G(t)$ of $f(t)e^{R(t)}$. We deduce that there exists a constant $C \in \mathbb{R}$ such that

$$e^{R(t)}u(t) = G(t) + C \Rightarrow u(t) = e^{-R(t)}G(t) + Ce^{-R(t)}.$$

Take for example the equation

$$u'(t) + 2tu(t) = t.$$

In this case

$$r(t) = 2t, \ \ f(t) = t.$$

We can choose $R(t) = t^2$ and we have

$$\frac{d}{dt}\big(e^{t^2}u(t) \big) = e^{t^2}u'(t) + 2te^{t^2}u(t) = e^{t^2}t,$$

so that

$$e^{t^2}u(t) = \int e^{t^2}tdt = \frac{1}{2}\int e^{t^2}d(t^2) = \frac{1}{2}e^{t^2} + C$$
$$\Rightarrow u(t) = e^{-t^2}\left(C + \frac{1}{2}e^{t^2} \right) = Ce^{-t^2} + \frac{1}{2}. \qquad \square$$

8.6. Exercises

Exercise 8.1. Let $n \in \mathbb{N}$, $x_0, c_0, c_1, \dots, c_n \in \mathbb{R}$ and

$$P(x) = c_0 + \frac{c_1}{1!}(x-x_0) + \frac{c_2}{2!}(x-x_0)^2 + \cdots + \frac{c_n}{n!}(x-x_0)^n = \sum_{k=0}^{n} \frac{c_k}{k!}(x-x_0)^k.$$

(a) Prove that for any $k = 0, 1, 2, \dots, n$ we have

$$P^{(k)}(x_0) = c_k.$$

(b) Prove that if $Q(x) = q_0 + q_1 x + \cdots q_n x^n$ is a polynomial of degree $\leq n$ such that

$$Q^{(k)}(x_0) = c_k, \;\; \forall k = 0, 1, 2, \dots, n,$$

then $Q(x) = P(x)$, $\forall x \in \mathbb{R}$.

Hint. Consider the difference $D(x) = P(x) - Q(x)$, observe that

$$D^{(k)}(x_0) = 0, \;\; \forall k = 0, 1, 2, \dots, n,$$

and conclude from the above that $D(x) = 0$, $\forall x \in \mathbb{R}$. To reach this conclusion write

$$D(x) = d_0 + d_1 x + \cdots + d_n x^n,$$

and observe first that $D^{(n)}(x) = n! d_n$, $\forall x \in \mathbb{R}$. □

Exercise 8.2. Suppose that $a, b \in \mathbb{R}$, $b \geq 0$ and consider $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{1 + ax^2}{1 + bx^2}.$$

Find the degree 4 Taylor polynomial of f at $x_0 = 0$. For which values of a, b does this polynomial coincide with the degree 4 Taylor polynomial of $\cos x$ at $x_0 = 0$?

Hint. To simplify the computations of the derivatives of f at 0 use the following trick. Let $N(x) = 1 + ax^2$ be the numerator of the fraction, $D(x) = 1 + bx^2$ be the denominator. Then

$$N(0) = D(0) = 1, \;\; N'(0) = D'(0) = 0, \;\; N''(0) = 2a, \;\; D''(0) = 2b, \tag{8.43}$$

$$N^{(k)}(x) = D^{(k)}(x) = 0, \;\; \forall k \geq 3, \;\; x \in \mathbb{R}. \tag{8.44}$$

We have

$$N(x) = D(x)f(x), \;\; N'(x) = D'(x)f(x) + D(x)f'(x),$$
$$N''(x) = D''(x)f(x) + 2D'(x)f'(x) + D(x)f''(x),$$
$$N^{(n)}(x) \stackrel{(7.35)}{=} \sum_{k=0}^{n} \binom{n}{k} D^{(k)}(x) f^{(n-k)}(x) \stackrel{(8.44)}{=} \sum_{k=0}^{2} \binom{n}{k} D^{(k)}(x) f^{(n-k)}(x)$$
$$= D(x)f^{(n)}(x) + nD'(x)f^{(n-1)}(x) + \frac{n(n-1)}{2} D''(x) f^{(n-2)}(x), \;\; \forall n > 2.$$

We deduce

$$f(0) = D(0)f(0) = N(0) = 1, \;\; f'(0) = D(0)f'(0) = N'(0) - D'(0)f(0) \stackrel{(8.43)}{=} 0,$$
$$f''(0) = D(0)f''(0) = N''(0) - 2D'(0)f'(0) - D''(0)f(0) \stackrel{(8.43)}{=} N''(0) - D''(0)f(0) = 2a - 2b,$$
$$f^{(n)}(0) = D(0)f^{(n)}(0) = N^{(n)}(0) - nD'(0)f^{(n-1)}(0) - \frac{n(n-1)}{2} D''(0) f^{(n-2)}(0)$$

$$\stackrel{(8.43)}{=} -bn(n-1)f^{(n-2)}(0), \quad n > 2. \qquad \square$$

Exercise 8.3. Use the inequality $2 < e < 3$ and the strategy outlined in Remark 8.6 to show that

$$\left| e^h - \left(1 + \frac{h}{1!} + \cdots + \frac{h^n}{n!}\right)\right| \leq \frac{3|h|^{n+1}}{(n+1)!}, \quad \forall |h| \leq 1. \qquad \square$$

Exercise 8.4. Using Example 8.7 as a guide, compute $\cos 1$ up to two decimals. $\square$

Exercise 8.5. Approximate $\sqrt[3]{8.1}$ using the degree 3 Taylor polynomial of $f(x) = \sqrt[3]{x}$ at $x_0 = 8$. Estimate the error of this approximation using the Lagrange estimate (8.3). $\square$

Exercise 8.6. Find the Taylor series of the function

$$f(x) = \frac{1}{1-x}, \quad x \neq 1$$

at $x_0 = 0$. For which values of x is this series convergent? $\square$

Exercise 8.7. Prove that the Taylor series of $\ln(1-x)$ at $x_0 = 0$ is

$$-\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

and then show that this series converges to $\ln(1-x)$ for any $x \in (-1, \frac{1}{2})$.

Hint. Use Corollary 8.5.[1] $\square$

Exercise 8.8. (a) Prove that the Taylor series of $\sin x$ at $x_0 = 0$,

$$\sum_{k \geq 0} (-1)^k \frac{x^{2k+1}}{(2k+1)!},$$

is absolutely convergent for any $x \in \mathbb{R}$ and its sum is $\sin x$. Show that the convergence is uniform on any interval $[-R, R]$.

(b) Prove that the Taylor series of $\cos x$ at $x_0 = 0$,

$$\sum_{k \geq 0} (-1)^k \frac{x^{2k}}{(2k)!}$$

is absolutely convergent and for any $x \in \mathbb{R}$ and its sum is $\cos x$. Show that the convergence is uniform on any interval $[-R, R]$.

Hint. Use Corollary 8.5. $\square$

Exercise 8.9. Find

$$\lim_{x \to \infty} x \left[\frac{1}{e} - \left(\frac{x}{x+1}\right)^x\right]. \qquad \square$$

[1] The Taylor series of $\ln(1-x)$ at $x_0 = 0$ converges to $\ln(1-x)$ *for all* $|x| < 1$. However, the Lagrange remainder formula is not strong enough to prove this. We need a different remainder formula (9.47) to prove this stronger statement. For details see Example 9.52.

Exercise 8.10. Using the fact that the function $\ln : (0, \infty) \to \mathbb{R}$ is concave prove *Young's inequality*: if $p, q \in (1, \infty)$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}, \;\; \forall x, y > 0. \tag{8.45}$$

□

Exercise 8.11. Suppose that $a < b$ are two real numbers and $f : (a, b) \to \mathbb{R}$ is a convex function.

(a) Prove that for any $x_1 < x_2 < x_3 \in (a, b)$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

Hint. Give a geometric interpretation to this statement and then think geometrically.

(b) Suppose that $x_0 \in (a, b)$. Prove that the one-sided limits

$$m_\pm(x_0) = \lim_{h \to 0\pm} \frac{f(x_0 + h) - f(x_0)}{h}$$

exist, are finite and $m_-(x_0) \leq m_+(x_0)$.

(c) Suppose $x_0 \in (a, b)$ and $m_\pm(x_0)$ are as above. Fix $m \in [m_-(x_0), m_+(x_0)]$. Show that

$$f(x) \geq f(x_0) + m(x - x_0), \;\; \forall x \in (a, b).$$

Can you give a geometric interpretation of this fact?

(d) Prove that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is convex. For $x_0 := 0$, compute the numbers $m_\pm(x_0)$ defined as in (b). □

Exercise 8.12. [2] Suppose that $f : [0, 1] \to [0, \infty)$ is a C^2-function satisfying the following additional properties.

(i) $f'(x) \geq 0$, $\forall x \in [0, 1]$.

(ii) $f''(x) > 0$, $\forall x \in (0, 1)$.

(iii) $f(1) = 1$, $f'(1) > 1$ and $f(0) > 0$.

Prove that the following hold.

(a) $f(x) \in [0, 1]$, $\forall x \in [0, 1]$.

(b) If $x_0 \in (0, 1)$ is a fixed point of f, i.e., $f(x_0) = x_0$, then $f'(x_0) < 1$.

Hint. Argue by contradiction. Use the Mean Value Theorem with the quotient

$$\frac{f(1) - f(x_0)}{1 - x_0}.$$

[2]The results in this exercise are particularly useful in probability theory in the investigation of the so called *branching processes*.

(c) The function f has a *unique* fixed point x_* located in the *open* interval $(0,1)$.

Hint. Argue by contradiction. Suppose that f has two fixed points $x_* < y_*$. in $(0,1)$. Use the Mean Value Theorem for the quotient

$$\frac{f(y_*) - f(x_*)}{y_* - x_*}$$

and reach a contradiction using (b).

(d) Fix $s \in (0,1)$ and consider the sequence (x_n) defined by the recurrence

$$x_0 = s, \;\; x_{n+1} = f(x_n), \;\; \forall n \geq 0.$$

Prove that

$$\lim_n x_n = x_*,$$

where x_* is the unique fixed point of f located in the interval $(0,1)$.

Hint. The sequence is bounded since it lies in $[0,1]$. Show that the sequence is monotone and the limit lies in $(0,1)$. □

Exercise 8.13. Prove that for any $n \in \mathbb{N}$ and any numbers $x_1, x_2, \dots, x_n \geq 0$ we have

$$\left(\frac{x_1 + \cdots + x_n}{n}\right)^2 \leq \frac{x_1^2 + \cdots + x_n^2}{n}.$$

Hint. Use the Cauchy-Schwartz inequality. □

Exercise 8.14. Consider the *Gauss bell*, i.e., the function

$$\gamma : \mathbb{R} \to \mathbb{R}, \;\; \gamma(x) = e^{-\frac{x^2}{2}}.$$

(a) Prove that for any $n \in \mathbb{N}$ there exists a polynomial $H_n(x)$ of degree n such that

$$\gamma^{(n)}(x) = (-1)^n H_n(x)\gamma(x).$$

(The polynomial $H_n(x)$ is called the *degree n Hermite polynomial.*)

(b) Prove that

$$H_{n+1}(x) = xH_n(x) - H_n'(x), \;\; \forall n \in \mathbb{N}.$$

(c) Compute $H_1(x)$, $H_2(x)$, $H_3(x)$.

(d) Find the intervals of convexity and concavity of $\gamma(x)$.

(e) Sketch the graph of the function $\gamma(x)$. □

Exercise 8.15. Consider the *hyperbolic functions*

$$\cosh, \sinh : \mathbb{R} \to \mathbb{R}, \;\; \cosh x = \frac{e^x + e^{-x}}{2}, \;\; \sinh(x) = \frac{e^x - e^{-x}}{2}, \;\; \forall x \in \mathbb{R}.$$

(cosh=hyperbolic cosine, sinh= hyperbolic sine)

(a) Prove that

$$\cosh' x = \sinh x, \;\; \sinh' x = \cosh x,$$
$$\cosh^2 x - \sinh^2 x = 1, \;\; \cosh^2 x + \sinh^2 x = \cosh(2x), \;\; \forall x \in \mathbb{R}.$$

(b) Find the Taylor series of $\cosh x$ and $\sinh x$ at $x_0 = 0$.

(c) Prove that the function sinh is bijective and then find its inverse.

(d) Sketch the graphs of cosh and sinh. □

Exercise 8.16. Compute

$$\int xe^{2x}dx, \quad \int xe^{2x}\cos x dx, \int xe^{2x}\sin x dx, \quad \int \sin^3 x \cos^2 x dx.$$ □

Exercise 8.17. Compute

$$\int \frac{1}{(4+x^2)^5}dx$$

by reducing it to the computation in Example 8.46(d). □

Exercise 8.18. Compute

$$\int (\cos x)^{11}dx.$$ □

Exercise 8.19. Using the strategy outlined in Example 8.50 find the function $u(t), v(t), f(t)$ satisfying the differential equations

$$u'(t) + 2u(t) = t, \;\; v'(t) - v(t) = \cos t,$$
$$f'(t) - (\tan t)f(t) = t, \;\; -\frac{\pi}{2} < t < \frac{\pi}{2}.$$ □

Exercise 8.20. Suppose that we are given a huge container containing 200 liters of pure water. In this container, starting at $t = 0$, we continuously add 10 liters of salted water per minute containing 1.5 grams of salt per liter and, at the same time, the container is leaking salt-water mixture at a constant rate of 10 liters per minute. Denote by $m(t)$ the amount of salt (in grams) contained in the mixture after t minutes from the start.

(a) Prove that $m(t)$ satisfies the differential equation

$$\frac{dm}{dt} = 15 - \frac{m(t)}{20}.$$

(b) Recalling that initially there was no salt in the water, i.e., $m(0) = 0$, find $m(t)$ for any $t > 0$. □

8.7. Exercises for extra-credit

Exercise* 8.1. Suppose that $f : (0, \infty) \to \mathbb{R}$ is a differentiable function such that

$$\lim_{x\to\infty} \big(f(x) + f'(x) \big) = 0.$$

Show that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0.$$ □

Exercise* 8.2. (a) Prove that for any $n \in \mathbb{N}$ and any real numbers $a, r > 0$ we have

$$a^{\frac{n}{n+1}} \leq \frac{1}{r}\left(\frac{r^{n+1}}{n+1} + \frac{na}{n+1} \right).$$

Hint: Use Young's inequality (8.45).

(b) Prove that if $\sum_{n\geq 0} a_n$ is a convergent series of positive numbers, then so is $\sum_{n\geq 0} a_n^{\frac{n}{n+1}}$. □

Exercise* 8.3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a C^3-function. Prove that there exists $a \in \mathbb{R}$ such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0. \qquad \square$$

Exercise* 8.4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a convex C^1 function. For $c \in \mathbb{R}$ we denote by $E_c(x)$ the function

$$E_c : \mathbb{R} \to \mathbb{R}, \;\; E_c(x) = \frac{x^2}{2} - cx + f(x).$$

(a) Prove that E_c has a unique critical point.

(b) Prove that the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + f'(x)$ is bijective. □

Exercise* 8.5. Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function satisfying

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}, \;\; \forall x, y \in [a, b].$$

Prove that f is convex. □

Exercise* 8.6. Suppose that $f : (a, b) \to \mathbb{R}$ is a convex function. Prove that f is continuous.

Hint. You need to use the facts proven in Exercise 8.11. □

Exercise* 8.7. Show that for any positive real numbers a, b, c we have

$$a + b + c \leq \frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab}. \qquad \square$$

Exercise* 8.8. Fix a natural number n and positive real numbers $x_1, \dots, x_n$. For any $\alpha > 0$ we set

$$M_\alpha(x_1, \dots, x_n) := \left(\frac{x_1^\alpha + \cdots + x_n^\alpha}{n}\right)^{\frac{1}{\alpha}}.$$

(a) Show that

$$M_\alpha(x_1, \dots, x_n) \leq M_\beta(x_1, \dots, x_n), \;\; \forall 0 < \alpha < \beta.$$

Hint. Use Hölder's inequality (8.21).

(b) Compute

$$\lim_{\alpha \to 0+} M_\alpha(x_1, \dots, x_n). \qquad \square$$

Exercise* 8.9. (a) Prove that for any $n \in \mathbb{N}$ the equation $x^n + x = 1$ has a unique positive solution x_n.

(b) Prove that

$$\lim_{n \to \infty} x_n = 1. \qquad \square$$

Chapter 9

Integral calculus

9.1. The integral as area: a first look

The Riemann integral is a very complicated infinite summation process that is often required when we want to compute areas or volumes of more irregular regions.

By way of motivation, let us consider a famous problem first solved by Archimedes by other means. Consider the arc of parabola in Figure 9.1 given by the equation

$$y = x^2, \;\; 0 \leq x \leq 1.$$

We would like to compute the area of the region R between the x-axis, the parabola and the vertical line $x = 1$.

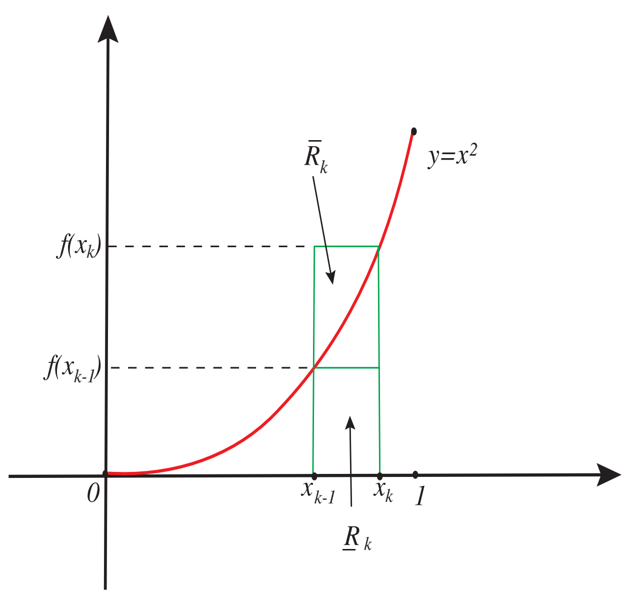

Figure 9.1. *Computing the area underneath an arc of parabola.*

To compute the area of R we subdivide the interval $[0,1]$ into N equal parts, where N is a very large natural number. We obtain the points

$$x_0 = 0,\ \ x_1 = \frac{1}{N},\ \ x_2 = \frac{2}{N}, \dots, x_N = \frac{N}{N}.$$

For each $k = 1, 2, \dots, N$ we denote by R_k the very thin slice of R of width $\frac{1}{N}$ delimited by the vertical lines $x = x_{k-1}$ and $x = x_k$. We have thus decomposed R into N thin slices $R_1, \dots, R_N$ and

$$\operatorname{area}(R) = \sum_{k=1}^{n} \operatorname{area}(R_k) = \operatorname{area}(R_1) + \cdots + \operatorname{area}(R_N).$$

Now observe that the slice R_k contains a thin rectangle $\underline{R}_k$ of height $f(x_{k-1})$ and is contained in a thin rectangle $\overline{R}_k$ of height $f(x_k)$; see Figure 9.1. Thus

$$f(x_{k-1}) \times (x_k - x_{k-1}) = \operatorname{area}(\underline{R}_k) \leq \operatorname{area}(R_k) \leq \operatorname{area}(\overline{R}_k) = f(x_k) \times (x_k - x_{k-1}).$$

Since $f(x_k) = \frac{k^2}{N^2}$ and $x_k - x_{k-1} = \frac{1}{N}$ we deduce

$$\frac{(k-1)^2}{N^3} \leq \operatorname{area}(R_k) \leq \frac{k^2}{N^3},$$

and thus

$$\underbrace{\sum_{k=1}^{N} \frac{(k-1)^2}{N^3}}_{=:L_N} \ \leq\ \underbrace{\sum_{k=1}^{N} \operatorname{area}(R_k)}_{=\operatorname{area}(R)} \ \leq\ \underbrace{\sum_{k=1}^{N} \frac{k^2}{N^3}}_{=:U_N}. \tag{9.1}$$

Thus

$$L_N \leq \operatorname{area}(R) \leq U_N. \tag{9.2}$$

Observe that

$$L_N = \frac{0^2}{N^3} + \frac{1^2}{N^3} + \cdots + \frac{(N-1)^2}{N^3} = \frac{1^2 + 2^2 + \cdots + (N-1)^2}{N^3},$$

$$U_N = \frac{1^2}{N^3} + \cdots + \frac{(N-1)^2}{N^3} + \frac{N^2}{N^3} = \frac{1^2 + 2^2 + \cdots + N^2}{N^3},$$

so that

$$U_N - L_N = \frac{N^2}{N^3} = \frac{1}{N}.$$

For N very large, the difference $U_N - L_N$ is very small and thus the sequence (L_N) converges if and only if the sequence (U_N) converges. Moreover, the inequality (9.2) shows that the common limit of these sequences, if it exists, must be equal to the area of R. To compute the limit of U_N we use the following famous identity whose proof is left to you as an exercise.

$$1^2 + 2^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}. \tag{9.3}$$

We deduce that

$$U_N = \frac{N(N+1)(2N+1)}{6N^3} = \frac{1}{6}\frac{N}{N}\frac{N+1}{N}\frac{2N+1}{N} \to \frac{2}{6} \ \text{ as } \ N \to \infty.$$

Thus

$$\operatorname{area}(R) = \frac{1}{3}.$$

This example describes the bare bones of the process called *integration*. As this simple example suggests, the integration it involves a sophisticated infinite summation and a bit of good fortune, in the guise of (9.3), that allowed us to actually compute the result of this infinite summation.

We will spend the rest of this chapter describing rigorously and in great generality this process and we will show that in a large number of cases we can cleverly create our good fortune and succeed in carrying out explicit computations of the limits of infinite summations involved.

9.2. The Riemann integral

The process sketched in the previous section can be carried out in greater generality. We present the quite involved details in this section.

Definition 9.1 (Partitions)**.** Fix an interval $[a, b]$, $a < b$.

(a) A *partition* $\boldsymbol{P}$ of $[a, b]$ is a finite collection of points $x_0, x_1, \dots, x_n$ of the interval such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The natural number n is called the *order* of the partition, while the points $x_0, \dots, x_n$ are called the *nodes* of the partition. The intervals

$$[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$$

are called the *intervals of the partition*. The interval $[x_{k-1}, x_k]$ is called the k-th interval of the partition and it is denoted by $I_k(\boldsymbol{P})$. Its length is denoted by $\Delta_k(\boldsymbol{P})$ or Δx_k. The largest of these lengths is called the *mesh size* of the partition and it is denoted by $\|\boldsymbol{P}\|$,

$$\|\boldsymbol{P}\| := \max_{1 \leq k \leq n} (x_k - x_{k-1}) = \max_{1 \leq k \leq n} \Delta_k(\boldsymbol{P}).$$

We denote by $\mathcal{P}_{[a,b]}$ the collection of all partitions of the interval $[a, b]$.

(b) A *sample* of a partition $\boldsymbol{P}$ of order n is a collection $\underline{\xi}$ consisting of n points $\xi_1, \dots, \xi_n$ such that

$$\xi_k \in I_k(\boldsymbol{P}), \;\; \forall k = 1, \dots, n.$$

The point ξ_k is called the *sample point* of the interval $I_k(\boldsymbol{P})$. We denote by $\mathcal{S}(\boldsymbol{P})$ the collection of all possible samples of the partition $\boldsymbol{P}$.

(c) A *sampled partition* of the interval $[a, b]$ is a pair $(\boldsymbol{P}, \underline{\xi})$, where $\boldsymbol{P}$ is a partition of $[a, b]$ and $\underline{\xi} \in \mathcal{S}(\boldsymbol{P})$ is a sample of $\boldsymbol{P}$. □

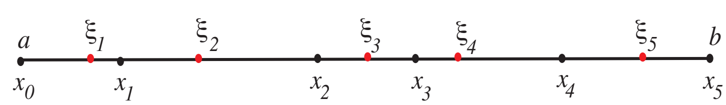

Figure 9.2. *A sampled partition of order* 5 *of an interval* $[a, b]$. *Its longest interval is* $[x_1 . x_2]$ *so its mesh size is* $(x_2 - x_1)$.

Example 9.2. Any compact interval $[a,b]$ has a natural partition $\boldsymbol{U}_n$ of order n corresponding to a subdivision of $[a,b]$ into n subintervals of order n. More precisely, $\boldsymbol{U}_n$ is defined by the points

$$x_0 = a,\ \ x_1 = a + \frac{1}{n}(b-a),\ \ x_k = a + \frac{k}{n}(b-a),\ \ k = 0,1,\dots,n.$$

The partition $\boldsymbol{U}_n$ is called the *uniform partition of order* n of $[a,b]$. Note that

$$\|\boldsymbol{U}_n\| = \frac{b-a}{n}. \qquad \square$$

Definition 9.3. Let $f : [a,b] \to \mathbb{R}$ be a function defined on the closed and bounded interval $[a,b]$. Given a partition $\boldsymbol{P} = (x_0 < \cdots < x_n)$ of $[a,b]$, and a sample $\underline{\xi}$ of $\boldsymbol{P}$, we define the *Riemann sum* of f associated to the sampled partition $(P,\underline{\xi})$ to be the number

$$\boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}) = \sum_{k=1}^{n} f(\xi_k)\Delta_k(\boldsymbol{P}) = \sum_{k=1}^{n} f(\xi_k)\Delta x_k = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}). \qquad \square$$

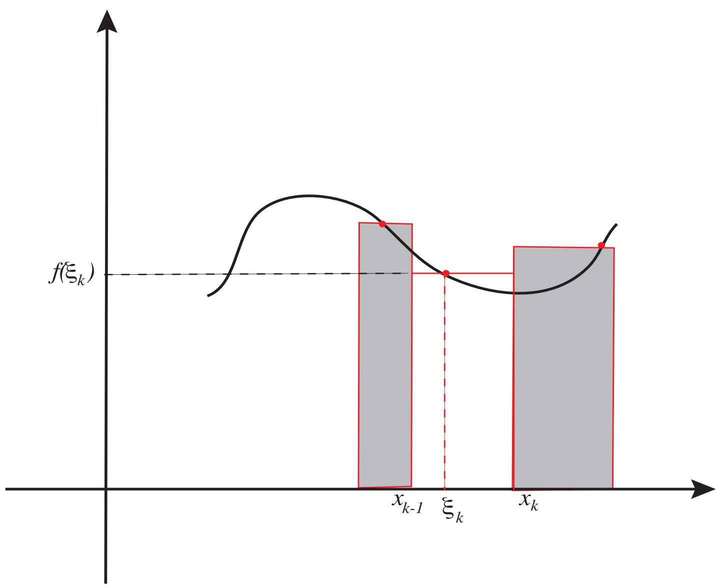

Figure 9.3. *The term $f(\xi_k)\Delta x_k$ is the area of a rectangle.*

As depicted in Figure 9.3, each term $f(\xi_k)(x_k - x_{k-1})$ in a Riemann sum is equal to the area of a "thin" rectangle of width $\Delta x_k = (x_k - x_{k-1})$, and height given by the altitude of the point on the graph of f determined by the sample point $\xi_k \in [x_{k-1}, x_k]$. The Riemann sum is therefore the area of the region formed by putting side by side each of these thin rectangles. The hope is that the area of this rather jagged looking region is an approximation for the area of the region under the graph of f. The next definition makes this intuition precise.

Definition 9.4. Suppose that $f : [a,b] \to \mathbb{R}$ is a function defined on the *closed and bounded* interval $[a,b]$. We say that f is *Riemann integrable on* $[a,b]$ if there exists a real number

S with the following property: for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that *for any partition* $\boldsymbol{P}$ of $[a,b]$ with mesh size $\|\boldsymbol{P}\| < \delta$ *and any sample* $\underline{\xi}$ of $\boldsymbol{P}$ we have

$$\left|\, S - \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) \,\right| < \varepsilon.$$

Equivalently, as a quantified statement, the above reads

$$\begin{aligned} &\exists S \in \mathbb{R},\ \ \forall \varepsilon > 0,\ \ \exists \delta = \delta(\varepsilon) > 0,\ \ \forall \boldsymbol{P} \in \mathcal{P}_{[a,b]},\ \ \forall \underline{\xi} \in \mathcal{S}(\boldsymbol{P}) : \\ &\qquad\qquad\qquad\qquad \|\boldsymbol{P}\| < \delta \Rightarrow \left|\, S - \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) \,\right| < \varepsilon. \end{aligned} \tag{9.4}$$

We will denote by $\mathcal{R}[a,b]$ the collection of all Riemann integrable functions $f : [a,b] \to \mathbb{R}$. □

Suppose that $f : [a,b] \to \mathbb{R}$ is Riemann integrable. For any $n \in \mathbb{N}$ we fix a sample $\underline{\xi}^{(n)}$ of $\boldsymbol{U}_n$, the uniform partition of order n of $[a,b]$. If S is *any* real number satisfying (9.4), then from the equality

$$\lim_{n\to\infty} \|\boldsymbol{U}_n\| = 0$$

we deduce that

$$S = \lim_{n\to\infty} \boldsymbol{S}\big(\, f, \boldsymbol{U}_n, \underline{\xi}^{(n)} \,\big).$$

Since a convergent sequence has a *unique* limit, we deduce that there exists precisely one real number I satisfying (9.4). This real number is called the *Riemann integral* of f on $[a,b]$ and it is denoted by

$$\int_a^b f(x)dx.$$

It bears repeating the definition of $\int_a^b f(x)dx$.

The Riemann integral of f over $[a,b]$, when it exists, is the ***unique real number*** $\int_a^b f(x)dx$ *with the following property: for any $\varepsilon > 0$ there exists $= \delta = \delta(\varepsilon) > 0$ such that for any partition $\boldsymbol{P}$ of $[a,b]$ with mesh $\|\boldsymbol{P}\| < \delta$, and for any sample $\underline{\xi}$ of $\boldsymbol{P}$, the Riemann sum $\boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi})$ is within ε of $\int_a^b f(x)dx$, i.e.,*

$$\left| \int_a^b f(x)dx - \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) \right| < \varepsilon.$$

We can loosely rephrase this as follows

$$\int_a^b f(x)dx = \lim_{\substack{\|\boldsymbol{P}\|\to 0,\\ \underline{\xi}\in\mathcal{S}(\boldsymbol{P})}} \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}). \tag{9.5}$$

Example 9.5. Consider the constant function $f : [a,b] \to \mathbb{R}$, $f(x) = C$, for all $x \in [a,b]$ where C is a fixed real number. Note that for any sampled partition of order n $(\boldsymbol{P}, \underline{\xi})$ of $[a,b]$ we have

$$\begin{aligned} \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) &= f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \cdots + f(\xi_n)(x_n - x_{n-1}) \\ &= C(x_1 - x_0) + C(x_2 - x_1) + \cdots + C(x_n - x_{n-1}) \\ &= C\Big(\, (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \,\Big) = C(x_n - x_0) = C(b - a). \end{aligned}$$

This shows that the constant function is integrable and

$$\int_a^b C dx = C(b-a). \qquad \square$$

It is natural to ask if there exist Riemann integrable functions more complicated than the constant functions. The next section will address precisely this issue. We will see that indeed, the world of integrable functions is very large. Until then, let us observe that not any function is Riemann integrable.

Proposition 9.6. *Suppose that $f:[a,b]\to\mathbb{R}$ is a Riemann integrable function. Then f is bounded, i.e.,*

$$-\infty < \inf_{x\in[a,b]} f(x) < \sup_{x\in[a,b]} f(x) < \infty.$$

Proof. We argue by contradiction. Suppose that $f:[a,b]\to\mathbb{R}$ is Riemann integrable and unbounded above, i.e.,

$$\sup_{x\in[a,b]} f(x) = \infty.$$

For any $n\in\mathbb{N}$ consider the uniform partition $\boldsymbol{U}_n$ of $[a,b]$. Then there exists $k=k(n)$ such that f is unbounded the interval $I_k = I_{k(n)}$ of this partition. For $j\neq k$ fix an arbitrary sample point $\xi_j\in I_j$. Since f is not bounded above on I_k, there exists $\xi_k\in I_k$ such that

$$f(\xi_k) > \frac{n}{\Delta x_k} - \sum_{j\neq k} f(\xi_j)\frac{\Delta x_j}{\Delta x_k} \Longleftrightarrow f(\xi_k)\Delta x_k + \sum_{j\neq k} f(\xi_j)\Delta x_j > n.$$

We obtain a sample $\underline{\xi}^{(n)}$ of $\boldsymbol{U}_n$ and for this sample we have

$$\boldsymbol{S}\big(f,\boldsymbol{U}_n,\underline{\xi}^{(n)}\big) = f(x_k)\Delta x_k + \sum_{j\neq k} f(\xi_j)\Delta x_j > n, \ \ \forall n\in\mathbb{N}.$$

The Riemann integrability of f implies that the sequence of Riemann sums $\boldsymbol{S}\big(f,\boldsymbol{U}_n,\underline{\xi}^{(n)}\big)$ is convergent. This contradicts the last inequality which states that this sequence is unbounded. $\square$

The above result shows that the function

$$f:[0,1]\to\mathbb{R}, \ \ f(x) = \begin{cases} 0, & x=0,\\ \frac{1}{\sqrt{x}}, & x\in(0,1], \end{cases}$$

is not Riemann integrable because it is not bounded.

9.3. Darboux sums and Riemann integrability

To be able to construct examples of integrable functions we need a criterion for recognizing such functions, more flexible than the definition. Fortunately there is one such criterion due to G. Darboux. To formulate it we need to introduce several new concepts.

Definition 9.7. Suppose that $f : [a, b] \to \mathbb{R}$ is a *bounded* function defined on the closed and bounded interval $[a, b]$. For any partition $\boldsymbol{P}$ of $[a, b]$ of order n we set

$$\boldsymbol{S}^*(f, \boldsymbol{P}) := \sum_{k=1}^{n} \sup_{x \in I_k(\boldsymbol{P})} f(x) \Delta x_k,$$

$$\boldsymbol{S}_*(f, \boldsymbol{P}) := \sum_{k=1}^{n} \inf_{x \in I_k(\boldsymbol{P})} f(x) \Delta x_k,$$

$$\omega(f, \boldsymbol{P}) := \sum_{k=1} \operatorname{osc}(f, I_k) \Delta x_k,$$

where

- $I_k = I_k(\boldsymbol{P})$ is the k-th interval of the partition $\boldsymbol{P}$,
- Δx_k is the length of I_k,
- $\operatorname{osc}(f, I_k)$ denotes the oscillation of f on I_k.

The quantity $\boldsymbol{S}^*(f, \boldsymbol{P})$ is called *the upper Darboux sum* of the function f determined by the partition $\boldsymbol{P}$, while $\boldsymbol{S}_*(f, \boldsymbol{P})$ is called *the lower Darboux sum* of the function f determined by the partition $\boldsymbol{P}$. We will refer to $\omega(f, \boldsymbol{P})$ as the *mean oscillation of f* along $\boldsymbol{P}$. □

Proposition 9.8. *If $f : [a, b] \to \mathbb{R}$ is a bounded function, then for any partition $\boldsymbol{P}$ of $[a, b]$ and any sample $\underline{\xi}$ of $\boldsymbol{P}$ we have*

$$\boldsymbol{S}_*(f, \boldsymbol{P}) \leq \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) \leq \boldsymbol{S}^*(f, \boldsymbol{P}), \tag{9.6a}$$

$$\omega(f, \boldsymbol{P}) = \boldsymbol{S}^*(f, \boldsymbol{P}) - \boldsymbol{S}_*(f, \boldsymbol{P}). \tag{9.6b}$$

Proof. Suppose that $\boldsymbol{P}$ is a partition of order n of $[a, b]$ and $\underline{\xi}$ is a sample of $\boldsymbol{P}$. For $k = 1, \ldots, n$ we denote by I_k the k-the interval of $\boldsymbol{P}$ and we set

$$M_k := \sup_{x \in I_k} f(x), \;\; m_k := \inf_{x \in I_k} f(x).$$

Then $M_k - m_k = \operatorname{osc}(f, I_k)$ and

$$\begin{aligned}\boldsymbol{S}^*(f, \boldsymbol{P}) - \boldsymbol{S}_*(f, \boldsymbol{P}) &= \big(M_1 \Delta x_1 + \cdots + M_n \Delta x_n \big) - \big(m_1 \Delta x_1 + \cdots + m_n \Delta x_n \big) \\ &= (M_1 - m_1) \Delta x_1 + \cdots + (M_n - m_n) \Delta x_n \\ &= \operatorname{osc}(f, I_1) \Delta x_1 + \cdots + \operatorname{osc}(f, I_n) \Delta x_n = \omega(f, \boldsymbol{P}).\end{aligned}$$

This proves (9.6b). If $\underline{\xi}$ is a sample of $\boldsymbol{P}$, then

$$m_k \Delta x_k \leq f(\xi_k) \Delta x_k \leq M_k \Delta x_k, \;\; \forall k = 1, \ldots, n,$$

so that

$$\sum_{k=1}^{n} m_k \Delta x_k \leq \sum_{k=1}^{n} f(\xi_k) \Delta x_k \leq \sum_{k=1}^{n} M_k \Delta x_k.$$

This proves (9.6a). □

Corollary 9.9. *If $f:[a,b]\to\mathbb{R}$ is a bounded function then for any partition P of $[a,b]$ and for any samples $\underline{\xi},\underline{\xi}'$ of $\boldsymbol{P}$ we have*

$$\left|\,\boldsymbol{S}(f,\boldsymbol{P},\underline{\xi})-\boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}')\,\right|\leq\omega(f,\boldsymbol{P}).$$

Proof. According to (9.6a) the Riemann sums $\boldsymbol{S}(f,\boldsymbol{P},\underline{\xi})$, $\boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}')$ are both contained in the interval $[\boldsymbol{S}_*(f,\boldsymbol{P}),\boldsymbol{S}^*(f,\boldsymbol{P})]$ so the distance between them must be smaller than the length of this interval which is equal to $\omega(f,\boldsymbol{P})$ according to (9.6b). □

Proposition 9.10. *Suppose that $f:[a,b]\to\mathbb{R}$ is a bounded function and $\boldsymbol{P}$ is a partition of $[a,b]$. If $\boldsymbol{P}'$ is a partition of $[a,b]$ obtained from $\boldsymbol{P}$ by adding one extra node x' in the interior of some interval of $\boldsymbol{P}$, then*

$$\boldsymbol{S}_*(f,\boldsymbol{P})\leq\boldsymbol{S}_*(f,\boldsymbol{P}')\leq\boldsymbol{S}^*(f,\boldsymbol{P}')\leq\boldsymbol{S}^*(f,\boldsymbol{P}).$$

Thus, by adding a node the upper Darboux sums decrease, while the lower Darboux sums increase.

Proof. The inequality (9.6a) shows that $\boldsymbol{S}_*(f,\boldsymbol{P}')\leq\boldsymbol{S}^*(f,\boldsymbol{P}')$. Suppose that the extra node x' is contained in (x_{k-1},x_k). We set

$$M_k:=\sup_{x\in I_k}f(x),\;\; m_k:=\inf_{x\in I_k}f(x).$$

Then

$$\begin{aligned}\boldsymbol{S}_*(f,\boldsymbol{P}')&=\sum_{j<k}m_j\Delta x_j+\underbrace{\inf_{x\in[x_{k-1},x']}f(x)}_{\geq m_k}(x'-x_{k-1})+\underbrace{\inf_{[x',x_k]}f(x)}_{\geq m_k}(x_k-x')+\sum_{\ell>k}m_\ell\Delta x_\ell\\&\geq\sum_{j<k}m_j\Delta x_j+\underbrace{m_k(x'-x_{k-1})+m_k(x_k-x')}_{=m_k(x_k-x_{k-1})}+\sum_{\ell>k}m_\ell\Delta x_\ell\\&=\sum_{j<k}m_j\Delta x_j+m_k\Delta x_k+\sum_{\ell>k}m_\ell\Delta x_\ell=\sum_{i=1}^n m_i\Delta x_i=\boldsymbol{S}_*(f,\boldsymbol{P}).\end{aligned}$$

The inequality

$$\boldsymbol{S}^*(f,\boldsymbol{P}')\leq\boldsymbol{S}^*(f,\boldsymbol{P})$$

is proved in a similar fashion.

□

Definition 9.11. Given two partitions $\boldsymbol{P},\boldsymbol{P}'$ of $[a,b]$, we say that $\boldsymbol{P}'$ is a *refinement* of $\boldsymbol{P}$, and we write this $\boldsymbol{P}'\succ\boldsymbol{P}$, if $\boldsymbol{P}'$ is obtained from $\boldsymbol{P}$ by adding a few more nodes. □

Since the addition of nodes increases lower Darboux sums and decreases upper Darboux sums we deduce the following result.

Proposition 9.12. *Suppose that $f:[a,b]\to\mathbb{R}$ is a bounded function and $\boldsymbol{P},\boldsymbol{P}'$ are partitions of $[a,b]$. If $\boldsymbol{P}'\succ\boldsymbol{P}$, then*

$$\boldsymbol{S}_*(f,\boldsymbol{P})\leq\boldsymbol{S}_*(f,\boldsymbol{P}')\leq\boldsymbol{S}^*(f,\boldsymbol{P}')\leq\boldsymbol{S}^*(f,\boldsymbol{P}).$$ □

Corollary 9.13. *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and $\boldsymbol{P}, \boldsymbol{P}'$ are partitions of $[a, b]$. If $\boldsymbol{P}' \succ \boldsymbol{P}$,*

$$\omega(f, \boldsymbol{P}') \leq \omega(f, \boldsymbol{P}). \tag{9.7}$$

Proof. From (9.8) we deduce

$$\boldsymbol{S}_*(f, \boldsymbol{P}) \leq \boldsymbol{S}_*(f, \boldsymbol{P}') \leq \boldsymbol{S}^*(f, \boldsymbol{P}') \leq \boldsymbol{S}^*(f, \boldsymbol{P}),$$

so that,

$$\omega(f, \boldsymbol{P}') = \boldsymbol{S}^*(f, \boldsymbol{P}') - \boldsymbol{S}_*(f, \boldsymbol{P}') \leq \boldsymbol{S}^*(f, \boldsymbol{P}) - \boldsymbol{S}_*(f, \boldsymbol{P}) = \omega(f, \boldsymbol{P}).$$

□

Given two partitions $\boldsymbol{P}, \boldsymbol{P}'$ of $[a, b]$ we denote by $\boldsymbol{P} \cup \boldsymbol{P}'$ the partition whose set of nodes is the union of the sets of nodes of the partitions $\boldsymbol{P}$ and $\boldsymbol{P}'$. Clearly $\boldsymbol{P} \cup \boldsymbol{P}'$ is a refinement of both $\boldsymbol{P}$ and $\boldsymbol{P}'$. From Proposition 9.12 we deduce the following important consequence.

Corollary 9.14. *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and $\boldsymbol{P}_0, \boldsymbol{P}_1$ are partitions of $[a, b]$. Then*

$$\boldsymbol{S}_*(f, \boldsymbol{P}_1) \leq \boldsymbol{S}_*(f, \boldsymbol{P}_0 \cup \boldsymbol{P}_1) \leq \boldsymbol{S}^*(f, \boldsymbol{P}_0 \cup \boldsymbol{P}_1) \leq \boldsymbol{S}^*(f, \boldsymbol{P}_0). \tag{9.8}$$

□

The above corollary shows that if $f : [a, b] \to \mathbb{R}$ is a bounded function, then the set

$$\left\{ \boldsymbol{S}^*(f, \boldsymbol{P});\ \ \boldsymbol{P} \in \mathcal{P}_{[a,b]} \right\}$$

is bounded below. Indeed, if we denote by $\boldsymbol{U}_1$ the uniform partition of order 1 of $[a, b]$, then (9.8) shows that

$$\boldsymbol{S}_*(f, \boldsymbol{U}_1) \leq \boldsymbol{S}^*(f, \boldsymbol{P}),\ \ \forall \boldsymbol{P} \in \mathcal{P}_{[a,b]}.$$

We set

$$\boldsymbol{S}^*(f) := \inf\left\{ \boldsymbol{S}^*(f, \boldsymbol{P});\ \ \boldsymbol{P} \in \mathcal{P}_{[a,b]} \right\}.$$

Similarly, the set

$$\left\{ \boldsymbol{S}_*(f, \boldsymbol{P});\ \ \boldsymbol{P} \in \mathcal{P}_{[a,b]} \right\}$$

is bounded above and we define

$$\boldsymbol{S}_*(f) := \sup\left\{ \boldsymbol{S}_*(f, \boldsymbol{P});\ \ \boldsymbol{P} \in \mathcal{P}_{[a,b]} \right\}.$$

Proposition 9.15. *If $f : [a, b] \to \mathbb{R}$ is a bounded function, then*

$$\boldsymbol{S}_*(f) \leq \boldsymbol{S}^*(f). \tag{9.9}$$

Proof. From (9.8) we deduce that $\forall \boldsymbol{P}_0, \boldsymbol{P}_1 \in \mathcal{P}_{[a,b]}$ we have

$$\boldsymbol{S}_*(f, \boldsymbol{P}_1) \leq \boldsymbol{S}^*(f, \boldsymbol{P}_0) \Rightarrow \boldsymbol{S}_*(f, \boldsymbol{P}_1) \leq \inf_{\boldsymbol{P}_0} \boldsymbol{S}^*(f, \boldsymbol{P}_0) = \boldsymbol{S}^*(f)$$

$$\Rightarrow \boldsymbol{S}_*(f) = \sup_{\boldsymbol{P}_1} \boldsymbol{S}_*(f, \boldsymbol{P}_1) \leq \boldsymbol{S}^*(f).$$

□

Definition 9.16. Let $f : [a,b] \to \mathbb{R}$ be a *bounded* function.

(a) The numbers $\boldsymbol{S}_*(f)$ and respectively $\boldsymbol{S}^*(f)$ are called the *lower* and respectively *upper Darboux integrals* of f.

(b) The function f is called *Darboux integrable* if $\boldsymbol{S}_*(f) = \boldsymbol{S}^*(f)$. □

Theorem 9.17 (Riemann-Darboux)**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a bounded function. Then the following statements are equivalent.*

(i) *The function f is Riemann integrable.*

(ii) *The function f is Darboux integrable, i.e., $\boldsymbol{S}_*(f) = \boldsymbol{S}^*(f)$.*

(iii) $\inf_{\boldsymbol{P}} \omega(f, \boldsymbol{P}) = 0$, *i.e.,*

$$\forall \varepsilon > 0, \;\; \exists \boldsymbol{P}_\varepsilon \in \mathcal{P}_{[a,b]} : \;\; \omega(f, \boldsymbol{P}_\varepsilon) < \varepsilon. \tag{$\boldsymbol{\omega}_0$}$$

(iv) $\lim_{\|\boldsymbol{P}\| \to 0} \omega(f, \boldsymbol{P}) = 0$, *i.e.,*

$$\forall \varepsilon > 0 \;\; \exists \delta = \delta(\varepsilon) > 0 \;\; \forall \boldsymbol{P} \in \mathcal{P}_{[a,b]} : \;\; \|\boldsymbol{P}\| < \delta \Rightarrow \omega(f, \boldsymbol{P}) < \varepsilon. \tag{$\boldsymbol{\omega}$}$$

Proof. We will prove these equivalences using the following logical successions

$$(iii) \Longleftrightarrow (ii), \;\; (iv) \;\Rightarrow\; (iii), \;\; (iv) \Longleftrightarrow (i), \;\; (iii) \;\Rightarrow\; (iv).$$

(iii) $\Rightarrow$ (ii). For any $\varepsilon > 0$ we can find a partition $\boldsymbol{P}_\varepsilon$ such that $\omega(f, \boldsymbol{P}_\varepsilon) < \varepsilon$. Now observe that

$$\boldsymbol{S}_*(f, \boldsymbol{P}_\varepsilon) \leq \boldsymbol{S}_*(f) \leq \boldsymbol{S}^*(f) \leq \boldsymbol{S}^*(f, \boldsymbol{P}_\varepsilon),$$

and

$$\boldsymbol{S}^*(f, \boldsymbol{P}_\varepsilon) - \boldsymbol{S}_*(f, \boldsymbol{P}_\varepsilon) = \omega(f, \boldsymbol{P}_\varepsilon) < \varepsilon.$$

Hence

$$0 \leq \boldsymbol{S}^*(f) - \boldsymbol{S}_*(f) \leq \boldsymbol{S}^*(f, \boldsymbol{P}_\varepsilon) - \boldsymbol{S}_*(f, \boldsymbol{P}_\varepsilon) < \varepsilon, \;\; \forall \varepsilon > 0,$$

so that

$$\boldsymbol{S}_*(f) = \boldsymbol{S}^*(f).$$

(ii) $\Rightarrow$ (iii). We know that $\boldsymbol{S}_*(f) = \boldsymbol{S}^*(f)$. Denote by $\boldsymbol{S}(f)$ this common value. Since

$$\boldsymbol{S}(f) = \boldsymbol{S}_*(f) = \sup_{\boldsymbol{P}} \boldsymbol{S}_*(f, \boldsymbol{P}),$$

we deduce that for any $\varepsilon > 0$ there exists a partition P_ε^- such that

$$\boldsymbol{S}(f) - \frac{\varepsilon}{2} < \boldsymbol{S}_*(f, \boldsymbol{P}_\varepsilon^-) \leq \boldsymbol{S}(f).$$

Since

$$\boldsymbol{S}(f) = \boldsymbol{S}^*(f) = \inf_{\boldsymbol{P}} \boldsymbol{S}^*(f, \boldsymbol{P}),$$

we deduce that for any $\varepsilon > 0$ there exists a partition P_ε^+ such that

$$\boldsymbol{S}(f) \leq \boldsymbol{S}^*(f, \boldsymbol{P}_\varepsilon^+) < \boldsymbol{S}(f) + \frac{\varepsilon}{2}.$$

Hence

$$\boldsymbol{S}(f) - \frac{\varepsilon}{2} < \boldsymbol{S}_*(f, \boldsymbol{P}_\varepsilon^-) \leq \boldsymbol{S}^*(f, \boldsymbol{P}_\varepsilon^+) < \boldsymbol{S}(f) + \frac{\varepsilon}{2}.$$

Now set $\boldsymbol{P}_\varepsilon := \boldsymbol{P}_\varepsilon^- \cup \boldsymbol{P}_\varepsilon^+$. We deduce from (9.8) that

$$\boldsymbol{S}(f) - \frac{\varepsilon}{2} < \boldsymbol{S}_*(f, \boldsymbol{P}_\varepsilon^-) \leq \boldsymbol{S}_*(f, \boldsymbol{P}_\varepsilon) \leq \boldsymbol{S}^*(f, \boldsymbol{P}_\varepsilon) \leq \boldsymbol{S}^*(f, \boldsymbol{P}_\varepsilon^+) < \boldsymbol{S}(f) + \frac{\varepsilon}{2}.$$

This proves that

$$\omega(f, \boldsymbol{P}_\varepsilon) = \boldsymbol{S}^*(f, \boldsymbol{P}_\varepsilon) - \boldsymbol{S}_*(f, \boldsymbol{P}_\varepsilon) < \varepsilon.$$

(iv) $\Rightarrow$ (iii). This is obvious.

(iv) $\Rightarrow$ (i). From the above we deduce that $\boldsymbol{S}_*(f) = \boldsymbol{S}^*(f)$. We set

$$\boldsymbol{S}(f) := \boldsymbol{S}_*(f) = \boldsymbol{S}^*(f).$$

We will show that f is integrable and its Riemann integral is $\boldsymbol{S}(f)$.

Fix $\varepsilon > 0$. According to ($\boldsymbol{\omega}$), there exists $\delta = \delta(\varepsilon) > 0$ such that for any partition $\boldsymbol{P}$ of $[a, b]$ satisfying $\|\boldsymbol{P}\| < \delta$ we have

$$\omega(f, \boldsymbol{P}) < \varepsilon.$$

Given a partition $\boldsymbol{P}$ such that $\|\boldsymbol{P}\| < \delta$ and $\underline{\xi}$ a sample of $\boldsymbol{P}$ we have

$$\boldsymbol{S}_*(f, \boldsymbol{P}) \leq \boldsymbol{S}(f) \leq \boldsymbol{S}^*(f, \boldsymbol{P}),$$
$$\boldsymbol{S}_*(f, \boldsymbol{P}) \leq \boldsymbol{S}(f, \boldsymbol{P}, \xi) \leq \boldsymbol{S}^*(f, \boldsymbol{P}).$$

Thus both numbers $\boldsymbol{S}(f)$ and $\boldsymbol{S}(f, \boldsymbol{P}, \xi)$ lie in the interval $[\boldsymbol{S}_*(f, \boldsymbol{P}), \boldsymbol{S}^*(f, \boldsymbol{P})]$ of length $\omega(f, \boldsymbol{P}) < \varepsilon$. Hence

$$\big|\, \boldsymbol{S}(f, \boldsymbol{P}, \xi) - \boldsymbol{S}(f) \,\big| < \varepsilon, \;\; \forall \|P\| < \delta(\varepsilon), \;\; \forall \underline{\xi} \in \mathcal{S}(\boldsymbol{P}).$$

This proves that f is Riemann integrable.

(i) $\Rightarrow$ (iv). We have to prove that if f is Riemann integrable, then f satisfies ($\boldsymbol{\omega}$). We first need an auxiliary result.

Lemma 9.18. *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function. Then for any $\varepsilon > 0$ and any partition $\boldsymbol{P}$ of $[a, b]$ there exist samples $\underline{\xi}'$ and $\underline{\xi}''$ of $\boldsymbol{P}$ such that*

$$\boldsymbol{S}_*(f, P) \leq \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}') < \boldsymbol{S}_*(f, P) + \varepsilon,$$
$$\boldsymbol{S}^*(f, \boldsymbol{P}) - \varepsilon < \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}'') \leq \boldsymbol{S}^*(f, \boldsymbol{P}).$$

Proof. We prove only the statement involving $\underline{\xi}'$. The proof of the existence of $\underline{\xi}''$ with the stated property is similar. Denote by n the order of $\boldsymbol{P}$ and by I_k the k-th interval of $\boldsymbol{P}$ and, as usual, we set

$$m_k = \inf_{x \in I_k} f(x).$$

In particular, there exists $\xi_k' \in I_k$ such that

$$m_k \leq f(\xi_k') < m_k + \frac{\varepsilon}{b - a}.$$

The collection $\underline{\xi}' = (\xi_k')_{1 \leq k \leq n}$ is a sample of $\boldsymbol{P}$ satisfying

$$m_k(x_k - x_{k-1}) \leq f(\xi_k')(x_k - x_{k-1}) < m_k(x_k - x_{k-1}) + \frac{\varepsilon}{b - a}(x_k - x_{k-1}).$$

Hence

$$\boldsymbol{S}_*(f, \boldsymbol{P}) = \sum_{k=1}^n m_k(x_k - x_{k-1}) \leq \underbrace{\sum_{k=1}^n f(\xi_k')(x_k - x_{k-1})}_{= \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}')}$$

$$< \underbrace{\sum_{k=1}^{n} m_k(x_k - x_{k-1})}_{=\boldsymbol{S}_*(f,\boldsymbol{P})} + \frac{\varepsilon}{b-a} \underbrace{\sum_{k=1}^{n}(x_k - x_{k-1})}_{=(b-a)} = \boldsymbol{S}_*(f,P) + \varepsilon.$$

□

We can now complete the proof of ($\boldsymbol{\omega}$). Since f is Riemann integrable, there exists $S_f \in \mathbb{R}$ such that, for any $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ with the property that for any partition $\boldsymbol{P}$ with mesh size $\|\boldsymbol{P}\| < \delta$ and any sample $\underline{\xi}$ of $\boldsymbol{P}$ we have

$$\big|\, S_f - \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}) \,\big| < \frac{\varepsilon}{4}. \tag{9.10}$$

According to Lemma 9.18 we can find samples $\underline{\xi}'$ and $\underline{\xi}''$ such that

$$\big|\, \boldsymbol{S}_*(f,\boldsymbol{P}) - \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}') \,\big|, \ \ \big|\, \boldsymbol{S}^*(f,\boldsymbol{P}) - \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}'') \,\big| < \frac{\varepsilon}{4}. \tag{9.11}$$

If $\|\boldsymbol{P}\| < \delta$, then

$$\omega(f,\boldsymbol{P}) = \big|\boldsymbol{S}^*(f,\boldsymbol{P}) - \boldsymbol{S}_*(f,\boldsymbol{P})\,\big|$$

$$\leq \big|\, \boldsymbol{S}_*(f,\boldsymbol{P}) - \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}') \,\big| + \big|\, \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}') - \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}'') \,\big| + \big|\, \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}'') - \boldsymbol{S}^*(f,\boldsymbol{P}) \,\big|$$

$$\stackrel{(9.11)}{<} \frac{\varepsilon}{4} + \big|\, \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}') - \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}'') \,\big| + \frac{\varepsilon}{4}$$

$$\leq \frac{\varepsilon}{2} + \big|\, \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}') - S_f \,\big| + \big|\, S_f - \boldsymbol{S}(f,\boldsymbol{P},\underline{\xi}'') \,\big| \stackrel{(9.10)}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

(iii) ⇒ (iv). We have to show that if f satisfies ($\boldsymbol{\omega}_0$), then it also satisfies ($\boldsymbol{\omega}$). We need the following auxiliary result.

Lemma 9.19. *Suppose that $\boldsymbol{P}_0 = \{a = z_0 < z_1 < \cdots < z_{n_0} = b\}$ is a partition of $[a,b]$ of order n_0. Denote by λ_0 the length of the shortest intervals of the partition $\boldsymbol{P}_0$, i.e.,*

$$\lambda_0 := \min_{1 \leq j \leq n_0} (z_j - z_{j-1}).$$

For any partition $\boldsymbol{P}$ such that $\|\boldsymbol{P}\| < \lambda_0$ we have

$$\omega(f,\boldsymbol{P}) \leq (n_0 - 1)\|\boldsymbol{P}\| \operatorname{osc}(f,[a,b]) + \omega(f,\boldsymbol{P}_0). \tag{9.12}$$

Proof. Denote by $I_1, \ldots, I_{n_0}$ the intervals of $\boldsymbol{P}_0$. Denote by n the order of $\boldsymbol{P}$, and by $J_1, \ldots, J_n$ the intervals of $\boldsymbol{P}$. We will denote by $\ell(J_k)$ the length of J_k and by $\ell(I_j)$ the length of I_j

Since $\ell(J_k) \leq \ell(I_j)$, $\forall j = 1, \ldots, n_0$, $k = 1, \ldots, n$ we deduce that the intervals J_k of $\boldsymbol{P}$ are of only the following two types.

Type 1. The interval J_k is contained in an interval I_j of $\boldsymbol{P}_0$.

Type 2. The interval J_k contains in the *interior* a node $z_{j(k)}$ of $\boldsymbol{P}_0$.

We denote by $\mathscr{J}^1$ the collection of Type 1 intervals of $\boldsymbol{P}$, and by $\mathscr{J}^2$ the collection of Type 2 intervals of $\boldsymbol{P}$. We remark that $\mathscr{J}^2$ could be empty. Moreover, for any node z_j of $\boldsymbol{P}_0$ there exists at most one Type 2 interval of $\boldsymbol{P}$ that contains z_j in the interior. Thus $\mathscr{J}^2$ consist of at most $n_0 - 1$ intervals, i.e., its cardinality $|\mathscr{J}^2|$ satisfies

$$|\mathscr{J}^2| \leq n_0 - 1.$$

We have

$$\omega(f,\boldsymbol{P})=\sum_{k=1}^{n}\operatorname{osc}(f,J_k)\ell(J_k)=\underbrace{\sum_{J_k\in\mathcal{J}^1}\operatorname{osc}(f,J_k)\ell(J_k)}_{=:S_1}+\underbrace{\sum_{J_k\in\mathcal{J}^2}\operatorname{osc}(f,J_k)\ell(J_k)}_{=:S_2}.$$

We now estimate S_1 from above

$$S_1=\sum_{j=1}^{n_0}\left(\sum_{J_k\subset I_j}\operatorname{osc}(f,J_k)\ell(J_k)\right)$$

($\operatorname{osc}(f,J_k)\leq\operatorname{osc}(f,I_j)$ whenever $J_k\subset I_j$)

$$\leq\sum_{j=1}^{n_0}\left(\sum_{J_k\subset I_j}\operatorname{osc}(f,I_j)\ell(J_k)\right)=\sum_{j=1}^{n_0}\operatorname{osc}(f,I_j)\underbrace{\left(\sum_{J_k\subset I_j}\ell(J_k)\right)}_{\leq\ell(I_j)}\leq\sum_{j=1}^{n_0}\operatorname{osc}(f,I_j)\ell(I_j)=\omega(f,\boldsymbol{P}_0).$$

Now observe that if J_k is a Type 2 interval of $\boldsymbol{P}$, then $\ell(J_k)\leq\|\boldsymbol{P}\|$ and $\operatorname{osc}(f,J_k)\leq\operatorname{osc}(f,[a,b])$. Hence

$$S_2\leq\sum_{J_k\in\mathcal{J}^2}\operatorname{osc}(f,[a,b])\|\boldsymbol{P}\|\leq|\mathcal{J}^2|\operatorname{osc}(f,[a,b])\|\boldsymbol{P}\|\leq(n_0-1)\operatorname{osc}(f,[a,b])\|\boldsymbol{P}\|.$$

Hence

$$\omega(f,\boldsymbol{P})=S_1+S_2\leq(n_0-1)\operatorname{osc}(f,[a,b])\|\boldsymbol{P}\|+\omega(f,\boldsymbol{P}_0).$$

□

Returning to our implication $(\boldsymbol{\omega}_0)\Rightarrow(\boldsymbol{\omega})$, we observe that $(\boldsymbol{\omega}_0)$ implies that for any $\varepsilon>0$ there exists a partition $\boldsymbol{P}_\varepsilon$ such that

$$\omega(f,\boldsymbol{P}_\varepsilon)<\frac{\varepsilon}{2}.$$

Denote by n_ε the order of $\boldsymbol{P}_\varepsilon$ and by $x_0<x_1<\cdots<x_{n_\varepsilon}$ the nodes of $\boldsymbol{P}_\varepsilon$. We set

$$\lambda_\varepsilon:=\min_{1\leq j\leq n_\varepsilon}(x_j-x_{j-1}).$$

Now choose $\delta=\delta(\varepsilon)>0$ such that

$$\delta<\lambda_\varepsilon\ \text{ and }\ (n_\varepsilon-1)\operatorname{osc}(f,[a,b])\delta<\frac{\varepsilon}{2}\iff\delta<\min\left(\lambda_\varepsilon,\ \frac{\varepsilon}{2(n_\varepsilon-1)\operatorname{osc}(f,[a,b])}\right).$$

If $\boldsymbol{P}$ is an arbitrary partition of $[a,b]$ such that $\|\boldsymbol{P}\|<\delta(\varepsilon)$, then Lemma 9.19 implies that

$$\omega(f,\boldsymbol{P})\leq(n_\varepsilon-1)\operatorname{osc}(f,[a,b])\delta+\omega(f,\boldsymbol{P}_\varepsilon)<\varepsilon.$$

This proves that f satisfies $(\boldsymbol{\omega})$ and completes the proof of the Riemann-Darboux Theorem.

□

We record here for later use a direct consequence of the above proof.

Corollary 9.20. *Suppose that $f:[a,b]\to\mathbb{R}$ is a Riemann integrable function. Then*

$$\int_a^b f(x)dx=\boldsymbol{S}_*(f)=\boldsymbol{S}^*(f). \tag{9.13}$$

In particular,

$$\boldsymbol{S}_*(f,\boldsymbol{P})\leq\int_a^b f(x)dx\leq\boldsymbol{S}^*(f,\boldsymbol{P}'),\ \ \forall\boldsymbol{P},\boldsymbol{P}'\in\mathcal{P}_{[a,b]}. \tag{9.14}$$

□

9.4. Examples of Riemann integrable functions

We are now going to collect the reward for the effort we spent proving the Riemann-Darboux theorem.

Proposition 9.21. *Any continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.*

Proof. We will use the Riemann-Darboux theorem to prove the claim. Note first that the Weierstrass Theorem 6.14 shows that f is bounded.

To prove that f satisfies ($\boldsymbol{\omega}$) we rely on the Uniform Continuity Theorem 6.29. According to this theorem, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any interval $I \subset [a, b]$ of length $< \delta$ we have

$$\operatorname{osc}(f, I) < \frac{\varepsilon}{b-a}.$$

If $\boldsymbol{P}$ is any partition of $[a, b]$ of order n and mesh size $\|\boldsymbol{P}\| < \delta(\varepsilon)$, then for any interval I_k of $\boldsymbol{P}$ we have

$$\operatorname{osc}(f, I_k) < \frac{\varepsilon}{b-a}.$$

Hence

$$\omega(f, \boldsymbol{P}) = \sum_{k=1}^{n} \operatorname{osc}(f, I_k)\Delta x_k < \frac{\varepsilon}{b-a} \underbrace{\sum_{k=1}^{n} \Delta x_k}_{=(b-a)} = \varepsilon.$$

This shows that f satisfies ($\boldsymbol{\omega}$) and thus it is Riemann integrable. □

Example 9.22. The function $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$ is continuous and thus integrable. Thus

$$\int_0^1 x^2 dx = \lim_{N\to\infty} \boldsymbol{S}_*(f, \boldsymbol{U}_N),$$

where $\boldsymbol{U}_N$ denote the uniform partition of order N of $[0, 1]$. Since f is nondecreasing we deduce that $\boldsymbol{S}_*(f, \boldsymbol{U}_N)$ coincides with the sum L_N defined in (9.1). As explained in Section 9.1 the sum L_N converges to $\frac{1}{3}$ as $N \to \infty$. □

Proposition 9.23. *Any nondecreasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.*

Proof. Clearly f is bounded since $f(a) \leq f(x) \leq f(b)$, $\forall x \in [a, b]$. If $\boldsymbol{P}$ is any partition of $[a, b]$ of order n, then for an interval $I_k = [x_{k-1}, x_k]$ of this partition we have

$$\operatorname{osc}(f, I_k) = f(x_k) - f(x_{k-1}),$$

$$\operatorname{osc}(f, I_k)\Delta x_k \leq \operatorname{osc}(f, I_k)\|\boldsymbol{P}\| = \|\boldsymbol{P}\|\big(\, f(x_k) - f(x_{k-1})\,\big)$$

so that

$$\omega(f, \boldsymbol{P}) = \sum_{k=1}^{n} \operatorname{osc}(f, I_k)\Delta x_k \leq \|\boldsymbol{P}\| \sum_{k=1}^{n} \big(\, f(x_k) - f(x_{k-1})\,\big) = \|\boldsymbol{P}\|\big(\, f(b) - f(a)\,\big).$$

This shows that f satisfies ($\boldsymbol{\omega}$) since

$$\lim_{\|\boldsymbol{P}\|\to 0} \|\boldsymbol{P}\|\big(\, f(b) - f(a)\,\big) = 0.$$

□

Proposition 9.24. *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function which is continuous on (a, b). Then f is Riemann integrable.*

Proof. We will prove that f satisfies ($\boldsymbol{\omega}_0$). Fix $\varepsilon > 0$ and choose a positive real number $d(\varepsilon)$ such that

$$\operatorname{osc}(f, [a, b])d(\varepsilon) < \frac{\varepsilon}{4}. \tag{9.15}$$

Denote by J_ε the compact interval $J_\varepsilon := [a + d(\varepsilon), b - d(\varepsilon)]$; see Figure 9.4.

The restriction of f to J_ε is continuous. The Uniform Continuity Theorem 6.29 implies that there exists $\delta = \delta(\varepsilon) < d(\varepsilon)$ with the property that for any interval $I \subset J_\varepsilon$ of length $\ell(I) < \delta(\varepsilon)$ we have

$$\operatorname{osc}(f, I) < \frac{\varepsilon}{2(b-a)}. \tag{9.16}$$

Consider a partition $\boldsymbol{P}_\varepsilon$ of order n of J_ε satisfying $\|\boldsymbol{P}\| < \delta(\varepsilon)$. We denote by I_k, $k = 1 \dots, n$, the intervals of $\boldsymbol{P}_\varepsilon$; see Figure 9.4. We set

$$I_* := [a, a + d(\varepsilon)], \quad I^* = [b - d(\varepsilon), b].$$

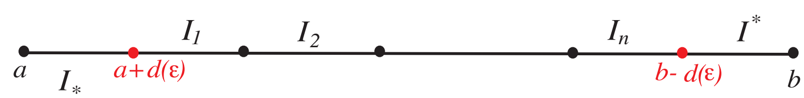

Figure 9.4. *Isolating the possible points of discontinuity of f.*

The collection of intervals

$$I_*, I_1, \dots, I_n, I^*$$

defines a partition $\widehat{\boldsymbol{P}}_\varepsilon$ of $[a, b]$; see Figure 9.4. We have

$$\omega(f, \widehat{\boldsymbol{P}}_\varepsilon) = \underbrace{\operatorname{osc}(f, I_*)\ell(I_*)}_{=:T_*} + \underbrace{\sum_{k=1}^n \operatorname{osc}(f, I_k)\ell(I_k)}_{=:T} + \underbrace{\operatorname{osc}(f, I^*)\ell(I^*)}_{=:T^*}.$$

Note that

$$\ell(I_*) = \ell(I^*) = d(\varepsilon).$$

so that

$$T_* = \operatorname{osc}(f, I_*)d(\varepsilon) \le \operatorname{osc}(f, [a, b])d(\varepsilon) \stackrel{(9.15)}{<} \frac{\varepsilon}{4},$$

$$T^* = \operatorname{osc}(f, I^*)d(\varepsilon) \le \operatorname{osc}(f, [a, b])d(\varepsilon) \stackrel{(9.15)}{<} \frac{\varepsilon}{4}.$$

Moreover,

$$T = \sum_{k=1}^n \operatorname{osc}(f, I_k)\ell(I_k) \stackrel{(9.16)}{<} \frac{\varepsilon}{2(b-a)} \sum_{k=1}^n \ell(I_k) = \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.$$

Hence,

$$\omega(f, \widehat{\boldsymbol{P}}_\varepsilon) = T_* + T + T^* < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

This proves that f satisfies ($\boldsymbol{\omega}_0$) and thus it is Riemann integrable. □

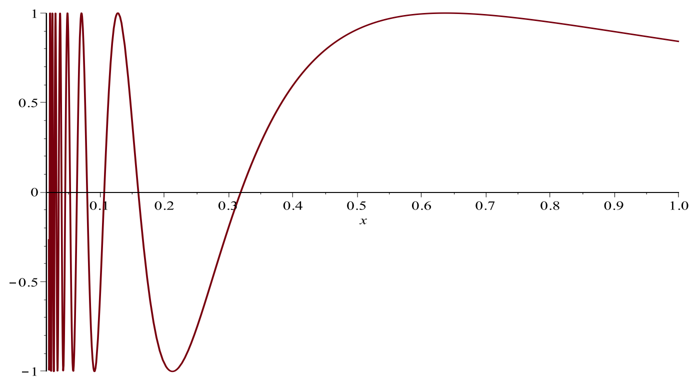

Figure 9.5. *A wildly oscillating, yet Riemann integrable function.*

Remark 9.25. Proposition 9.24 has some surprising nontrivial consequences. For example, it shows that the wildly oscillating function (see Figure 9.5)

$$f:[0,1]\to\mathbb{R},\quad f(x)=\begin{cases}\sin\Big(\frac{1}{x}\Big), & x\in(0,1],\\ 0, & x=0,\end{cases}$$

is Riemann integrable. □

Proposition 9.26. *Suppose that $f:[a,b]\to\mathbb{R}$ is a bounded function and $c\in(a,b)$. The following statements are equivalent.*

(i) *The function f is Riemann integrable on $[a,b]$.*

(ii) *The restrictions of $f|_{[a,c]}$ and $f|_{[c,b]}$ of f to $[a,c]$ and $[c,b]$ are Riemann integrable functions.*

Moreover, if f satisfies either one of the two equivalent conditions above, then

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx. \tag{9.17}$$

Proof. (i) $\Rightarrow$ (ii). Suppose that f is Riemann integrable on $[a,b]$. Given a partition $\boldsymbol{P}'$ of $[a,c]$ and a partition $\boldsymbol{P}''$ of $[c,b]$ we obtain a partition $\boldsymbol{P}'*\boldsymbol{P}''$ of $[a,b]$ whose set of nodes is the union of the sets of nodes of $\boldsymbol{P}'$ and $\boldsymbol{P}''$. Note that

$$\|\boldsymbol{P}'*\boldsymbol{P}''\|\leq\max\{\,\|\boldsymbol{P}'\|,\ \|\boldsymbol{P}''\|\,\},$$

and

$$\omega(\,f,\boldsymbol{P}'*\boldsymbol{P}''\,)=\omega(\,f|_{[a,c]},\boldsymbol{P}'\,)+\omega\big(\,f|_{[c,b]},\boldsymbol{P}'\,\big).$$

Since f is Riemann integrable on $[a,b]$, it satisfies the property ($\boldsymbol{\omega}$) so, for any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that, for any partition $\boldsymbol{P}$ of $[a,b]$ with mesh size $\|\boldsymbol{P}\|<\delta(\varepsilon)$, we have

$$\omega(f,\boldsymbol{P})<\varepsilon.$$

If the partitions $\boldsymbol{P}'$ and $\boldsymbol{P}''$ satisfy

$$\max\{\,\|\boldsymbol{P}'\|,\ \|\boldsymbol{P}''\|\,\}<\delta(\varepsilon),$$

then $\|\boldsymbol{P}' * \boldsymbol{P}''\| < \delta(\varepsilon)$ so that

$$\omega(f|_{[a,c]}, \boldsymbol{P}') + \omega(f|_{[c,b]}, \boldsymbol{P}'') = \omega(f, \boldsymbol{P}' * \boldsymbol{P}'') < \varepsilon.$$

This shows that both restrictions $f|_{[a,c]}$ and $f|_{[c,b]}$ satisfy ($\boldsymbol{\omega}$) and thus are Riemann integrable.

(ii) $\Rightarrow$ (i). We will prove that if $f|_{[a,c]}$ and $f|_{[c,b]}$ are Riemann integrable, then f is integrable on $[a,b]$. We invoke Theorem 9.17. It suffices to show that f satisfies ($\boldsymbol{\omega}_0$). Fix $\varepsilon > 0$. We have to prove that there exists a partition $\boldsymbol{P}_\varepsilon$ of $[a,b]$ such that $\omega(f, \boldsymbol{P}_\varepsilon) < \varepsilon$.

Since $f|_{[a,c]}$ and $f|_{[c,b]}$ are Riemann integrable, they satisfy ($\boldsymbol{\omega}_0$), and we deduce that there exist partitions $\boldsymbol{P}'_\varepsilon$ of $[a,c]$, and $\boldsymbol{P}''_\varepsilon$ of $[c,b]$ such that

$$\omega(f, \boldsymbol{P}'_\varepsilon), \;\; \omega(f, \boldsymbol{P}''_\varepsilon) < \frac{\varepsilon}{2}.$$

Then $\boldsymbol{P}_\varepsilon = \boldsymbol{P}'_\varepsilon * \boldsymbol{P}''_\varepsilon$ is a partition of $[a,b]$, and

$$\omega(f, \boldsymbol{P}_\varepsilon) = \omega(f, \boldsymbol{P}'_\varepsilon) + \omega(f, \boldsymbol{P}''_\varepsilon) < \varepsilon.$$

To prove (9.17) assume that f satisfies both (i) and (ii). Denote by $\boldsymbol{U}'_n$ the uniform partition of order n of $[a,c]$ and by $\boldsymbol{U}''_n$ the uniform partition of order n of $[c,b]$. Set

$$\boldsymbol{P}_n := \boldsymbol{U}'_n * \boldsymbol{U}''_n.$$

Note that

$$\|\boldsymbol{P}_n\| = \max\big(\|\boldsymbol{U}'_n\|, \; \|\boldsymbol{U}''_n\| \big) \to 0 \;\text{ as } n \to \infty. \tag{9.18}$$

Denote by $\underline{\xi}'_n$ the midpoint sample of $\boldsymbol{U}'_n$, and by $\underline{\xi}''_n$ the midpoint sample of $\boldsymbol{U}''_n$. Then $\underline{\xi}_n := \underline{\xi}_n \cup \underline{\xi}''_n$ is the midpoint sample of $\boldsymbol{P}_n$. We have

$$\boldsymbol{S}(f, \boldsymbol{P}_n, \underline{\xi}_n) = \boldsymbol{S}(f, \boldsymbol{U}'_n, \underline{\xi}'_n) + \boldsymbol{S}(f, \boldsymbol{U}''_n, \underline{\xi}''_n). \tag{9.19}$$

From (i), (9.18), and (9.5) we deduce that

$$\lim_{n\to\infty} \boldsymbol{S}(f, \boldsymbol{P}_n, \underline{\xi}_n) = \int_a^b f(x)dx.$$

From (ii), (9.18), and (9.5) we deduce that

$$\lim_{n\to\infty} \boldsymbol{S}(f, \boldsymbol{U}'_n, \underline{\xi}'_n) = \int_a^c f(x)dx,$$

$$\lim_{n\to\infty} \boldsymbol{S}(f, \boldsymbol{U}''_n, \underline{\xi}''_n) = \int_c^b f(x)dx.$$

The equality (9.17) now follows from the above three equalities after letting $n \to \infty$ in (9.19). □

Applying Proposition 9.26 iteratively we deduce the following consequence.

Corollary 9.27. *Suppose that* $f : [a,b] \to \mathbb{R}$ *is a bounded function and*

$$\boldsymbol{P} = (a = x_0 < x_1 < \cdots < x_n = b)$$

is a partition of $[a,b]$*. Then the following statements are equivalent.*

(i) *The function* f *is Riemann integrable on* $[a,b]$*.*

(ii) *For any* $k = 1, \ldots, n$ *the restriction of* f *to* $[x_{k-1}, x_k]$ *is Riemann integrable.*

Moreover, if any of the above two equivalent conditions is satisfied, then

$$\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^b f(x)dx. \tag{9.20}$$

□

Corollary 9.28. *If $f : [a,b] \to \mathbb{R}$ is a bounded function and $D \subset [a,b]$ is a finite set such that f is continuous at any point in $[a,b] \setminus D$, then f is Riemann integrable.*

Proof. We add to D the endpoints a, b if they are not contained in D and we obtain a partition $\boldsymbol{P}$ of $[a,b]$ such that f is continuous in the *interior* of any interval $[x_{k-1}, x_k]$ of $\boldsymbol{P}$. Proposition 9.24 implies that f is Riemann integrable on each of the intervals $[x_{k-1}, x_k]$ and Corollary 9.27 implies that f is integrable on $[a,b]$. □

Proposition 9.29. *If $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable, then for any constants $\alpha, \beta \in \mathbb{R}$ the sum $\alpha f + \beta g : [a,b] \to \mathbb{R}$ is also Riemann integrable and*

$$\int_a^b \big(\, \alpha f(x) + \beta g(x) \,\big) dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx. \tag{9.21}$$

Proof. We will show that f satisfies the definition of Riemann integrability, Definition 9.4. Observe first that if $(\boldsymbol{P}, \underline{\xi})$ is a sampled partition of $[a,b]$, then

$$\boldsymbol{S}\big(\, \alpha, f + \beta g, \boldsymbol{P}, \xi\big) = \alpha \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) + \beta \boldsymbol{S}(g, \boldsymbol{P}, \underline{\xi}). \tag{9.22}$$

Indeed, if the partition $\boldsymbol{P}$ is

$$\boldsymbol{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\},$$

and the sample $\underline{\xi}$ is $\underline{\xi} = (\xi_k)_{1 \leq k \leq n}$, then

$$\begin{aligned}\boldsymbol{S}\big(\, \alpha, f + \beta g, \boldsymbol{P}, \xi\big) &= \sum_k \big(\, \alpha f(\xi_k) + \beta g(\xi_k) \,\big) \Delta x_k = \sum_k \alpha f(\xi_k) \Delta x_k + \sum_k \beta g(\xi_k) \Delta x_k \\ &= \alpha \sum_k f(\xi_k) \Delta x_k + \beta \sum_k g(\xi_k) \Delta x_k = \alpha \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) + \beta \boldsymbol{S}(g, \boldsymbol{P}, \underline{\xi}).\end{aligned}$$

Set

$$K := (|\alpha| + |\beta| + 1).$$

Fix $\varepsilon > 0$. Since f is Riemann integrable, there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that, $\forall \boldsymbol{P} \in \mathcal{P}_{[a,b]}$, $\forall \underline{\xi} \in \mathcal{S}(\boldsymbol{P})$ we have

$$\|\boldsymbol{P}\| < \delta_1 \Rightarrow \left| \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) - \int_a^b f(x)dx \right| < \frac{\varepsilon}{K}. \tag{9.23}$$

Since g is Riemann integrable, there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that, $\forall \boldsymbol{P} \in \mathcal{P}_{[a,b]}$, $\forall \underline{\xi} \in \mathcal{S}(\boldsymbol{P})$ we have

$$\|\boldsymbol{P}\| < \delta_2 \Rightarrow \left| \boldsymbol{S}(g, \boldsymbol{P}, \underline{\xi}) - \int_a^b g(x)dx \right| < \frac{\varepsilon}{K}. \tag{9.24}$$

Set

$$\delta = \delta(\varepsilon) := \min\big(\, \delta_1(\varepsilon), \delta_2(\varepsilon) \,\big), \;\; S := \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

Let $\boldsymbol{P} \in \mathcal{P}_{[a,b]}$ be an arbitrary partition such that $\|\boldsymbol{P}\| < \delta$. Then for any sample $\underline{\xi} \in \mathcal{S}(\boldsymbol{P})$ we have

$$|\boldsymbol{S}(\alpha f + \beta g, \boldsymbol{P}, \xi) - S| \stackrel{(9.22)}{=} \left| \alpha\Big(\boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) - \int_a^b f(x)dx \Big) + \beta\Big(\boldsymbol{S}(g, \boldsymbol{P}, \underline{\xi}) - \int_a^b g(x)dx \Big) \right|$$

$$\leq |\alpha| \cdot \left| \boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi}) - \int_a^b f(x)dx \right| + |\beta| \cdot \left| \boldsymbol{S}(g, \boldsymbol{P}, \underline{\xi}) - \int_a^b g(x)dx \right|$$

(use (9.23) and (9.24))

$$\leq |\alpha| \frac{\varepsilon}{K} + |\beta| \frac{\varepsilon}{K} = \frac{|\alpha| + |\beta|}{|\alpha| + |\beta| + 1} \varepsilon < \varepsilon.$$

This proves that $\alpha f + \beta g$ is Riemann integrable and

$$\int_a^b f(x)dx = S = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

□

Corollary 9.30. *Suppose that $f, g : [a, b] \to \mathbb{R}$ are two functions such that*

$$f(x) = g(x), \quad \forall x \in (a, b).$$

If f is Riemann integrable, then so is g and, moreover,

$$\int_a^b f(x)dx = \int_a^b g(x)dx. \tag{9.25}$$

Proof. Consider the difference $h : [a, b] \to \mathbb{R}$, $h(x) = g(x) - f(x)$, $\forall x \in [a, b]$. Note that h is bounded on $[a, b]$ and continuous on (a, b) because $h(x) = 0$, $\forall x \in (a, b)$. Using Proposition 9.24 we deduce that h is Riemann integrable on $[a, b]$. Since $g = f + h$, we deduce from Proposition 9.29 that g is Riemann integrable on $[a, b]$ and

$$\int_a^b g(x)dx = \int_a^b f(x)dx + \int_a^b h(x)dx.$$

Thus, to prove (9.25) we have to show that

$$\int_a^b h(x)dx = 0.$$

To do this, denote by $\boldsymbol{U}_n$ the uniform partition of order n of $[a, b]$, and denote by $\underline{\xi}^{(n)}$ the sample of $\boldsymbol{U}_n$ consisting of the midpoints of the intervals of $\boldsymbol{U}_n$. Then

$$\boldsymbol{S}(h, \boldsymbol{U}_n, \underline{\xi}^{(n)}) = 0.$$

Since h is Riemann integrable, we have

$$\int_a^b h(x)dx = \lim_{n \to \infty} \boldsymbol{S}(h, \boldsymbol{U}_n, \underline{\xi}^{(n)}) = 0.$$

□

Example 9.31. We say that a function $f : [a, b] \to \mathbb{R}$ is *piecewise constant* if there exists a partition

$$\boldsymbol{P} = (a = x_0 < x_1 < \cdots < x_n = b)$$

and constants $c_1, \dots, c_n$ such that for any $k = 1, \dots, n$ the restriction of f to the open interval (x_{k-1}, x_k) is the constant function c_k. From the above corollary we deduce that f is Riemann integrable on each of the intervals $[x_{k-1}, x_k]$. Moreover, the computation in Example 9.5 implies that

$$\int_{x_{k-1}}^{x_k} f(t)dt = c_k(x_k - x_{k-1}).$$

Corollary 9.27 implies that f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)dx = c_1(x_1 - x_0) + \cdots + c_n(x_n - x_{n-1}).$$

□

Proposition 9.32. *Suppose that $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function, J is an interval containing the range of f and $G : J \to \mathbb{R}$ is a Lipschitz function. Then $G \circ f : [a, b] \to \mathbb{R}$ is Riemann integrable.*

Proof. Fix a positive constant L such that

$$|G(y_1) - G(y_2)| \leq L|y_1 - y_2|, \;\; \forall y_1, y_2 \in J.$$

Observe that for any $X \subset [a, b]$ and any $x', x'' \in X$ we have

$$\big|\, G \circ f(x') - G \circ f(x'') \,\big| \leq L|f(x') - f(x'')|.$$

Hence

$$\operatorname{osc}(G \circ f, X) = \sum_{x', x'' \in X} \big|\, G \circ f(x') - G \circ f(x'') \,\big| \leq L \sum_{x', x'' \in X} |f(x') - f(x'')| = L \operatorname{osc}(f, X).$$

We deduce as in the proof of Proposition 9.29 that for any partition $\boldsymbol{P}$ of $[a, b]$ we have

$$\omega(G \circ f, \boldsymbol{P}) \leq L\omega(f, \boldsymbol{P}).$$

Since f is Riemann integrable we deduce that

$$\lim_{\|\boldsymbol{P}\| \to 0} \omega(f, \boldsymbol{P}) = 0$$

so that

$$\lim_{\|\boldsymbol{P}\| \to 0} \omega(G \circ f, \boldsymbol{P}) = 0.$$

□

Corollary 9.33. *Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Then f^2 is also Riemann integrable on $[a, b]$.*

Proof. Since f is Riemann integrable it is bounded so its range is contained in some interval $[-M, M]$, $M > 0$. The function $G : [-M, M] \to \mathbb{R}$, $G(x) = x^2$ is Lipschitz on this interval because for any $x, y \in [-M, M]$ we have

$$|G(x) - G(y)| = |x^2 - y^2| = |x + y| \cdot |x - y| \leq (|x| + |y|)|x - y| \leq 2M|x - y|.$$

Proposition 9.32 implies that $G \circ f = f^2$ is Riemann integrable. □

Corollary 9.34. *If $f, g : [a, b] \to \mathbb{R}$ are Riemann integrable, then so is their product fg.*

Proof. The function $f + g$ is integrable according to Proposition 9.29. Invoking Corollary 9.33 we deduce that the functions $(f + g)^2, f^2, g^2$ are Riemann integrable. Proposition 9.29 now implies that the function

$$\frac{1}{2}\Big((f+g)^2 - f^2 - g^2 \Big) = \frac{1}{2}\big(f^2 + g^2 + 2fg - f^2 - g^2 \big) = fg$$

is Riemann integrable. □

Corollary 9.35. *Suppose that $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Then the function $|f|$ is also Riemann integrable.*

Proof. The function $G : \mathbb{R} \to \mathbb{R}$, $G(y) = |y|$ is Lipschitz so the function $G \circ f = |f|$ is Riemann integrable. □

☞ ***A very useful convention.*** We denoted the Riemann integral of a function $f : [a, b] \to \mathbb{R}$ with the symbol

$$\int_a^b f(x)dx,$$

where the lower endpoint a is at the bottom of the integral sign $\int$ and the upper endpoint b is at the top of the integral sign. We define

$$\int_b^a f(x)dx := -\int_a^b f(x)dx, \quad \int_a^a f(x)dx = 0.$$

There are several arguments in favor of this convention. For example, we can rewrite (9.30) as

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{a-b}\int_b^a f(x)dx. \tag{9.26}$$

This formulation will be especially useful when we do not know whether $a < b$ or $b < a$. The above equality says that it does not matter.

Another advantage comes from the following additivity identity.

$$\int_a^c f(x)dx = \int_a^{b} f(x)dx + \int_{b}^c f(x)dx, \quad \forall a, b, c \in \mathbb{R}. \tag{9.27}$$

If $a < b < c$, then (9.27) is an immediate consequence of Corollary 9.27. When the numbers a, b, c are situated in a different order, the identity (9.27) is still a consequence of Corollary 9.27, but in a more roundabout way. For example, if $a = 0$, $b = 2$ and $c = 1$, then

$$\int_0^1 f(x)dx = \int_0^2 f(x)dx - \int_1^2 f(x)dx = \int_0^{2} f(x)dx + \int_{2}^1 f(x)dx.$$ □

9.5. Basic properties of the Riemann integral

Now that we have seen how the concept of integrability interacts with the basic arithmetic operations on functions we want to discuss a few simple techniques for estimating Riemann integrals. All these techniques are based on the following simple result.

Proposition 9.36. *Suppose that $f : [a,b] \to \mathbb{R}$ is Riemann integrable and $f(x) \geq 0$ for any $x \in [a,b]$. Then*

$$\int_a^b f(x)dx \geq 0.$$

Proof. Denote by $\boldsymbol{U}_1$ the partition of $[a,b]$ consisting of a single interval. Then

$$0 \leq \big(\inf_{x\in[a,b]} f(x) \big)(b-a) = \boldsymbol{S}_*(f, \boldsymbol{U}_1) \overset{(9.14)}{\leq} \int_a^b f(x)dx.$$

□

Corollary 9.37. *If $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable functions and $f(x) \leq g(x)$, $\forall x \in [a,b]$, then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Proof. The function $(g - f)$ is integrable and nonnegative so

$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g(x) - f(x))dx \geq 0.$$

□

Corollary 9.38. *If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then*

$$\boxed{\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\, dx.} \tag{9.28}$$

Proof. We know that

$$f(x) \leq |f(x)| \;\text{ and }\; -f(x) \leq |f(x)|, \;\; \forall x \in [a,b].$$

Hence

$$\int_a^b f(x)dx \leq \int_a^b |f(x)|dx \;\text{ and }\; -\int_a^b f(x)dx \leq \int_a^b |f(x)|dx.$$

The last two inequalities imply (9.28).

□

Corollary 9.39. *Suppose that $f : [a,b] \to \mathbb{R}$ is a Riemann integrable function. We set*

$$m := \inf_{x\in[a,b]} f(x), \;\; M = \sup_{x\in[a,b]} f(x).$$

Then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Proof. We have

$$m \leq f(x) \leq M, \quad \forall x \in [a,b],$$

so that

$$m(b-a) = \int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx = M(b-a).$$

□

Definition 9.40. If $f : [a,b] \to \mathbb{R}$ is a Riemann integrable function, then the quantity

$$\frac{1}{b-a}\int_a^b f(x)dx$$

is called the *average value* of f, or the *mean* of f, or the *expectation* of f and we denote it by $\mathrm{Mean}(f)$. □

We see that we can rephrase the inequality in Corollary 9.39 as

$$\inf_{x \in [a,b]} f(x) \leq \mathrm{Mean}(f) \leq \sup_{x \in [a,b]} f(x). \tag{9.29}$$

Theorem 9.41 (Integral Mean Value Theorem)**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a **continuous** function. Then there exists $\xi \in [a,b]$ such that*

$$f(\xi) = \mathrm{Mean}(f),$$

i.e.,

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{9.30}$$

Proof. Let

$$m := \inf_{x \in [a,b]} f(x), \quad M = \sup_{x \in [a,b]} f(x).$$

Then (9.29) implies that $\mathrm{Mean}(f) \in [m, M]$.

On the other hand, since f is continuous we deduce from Weierstrass' Theorem 6.14 that there exist $x_*, x^* \in [a,b]$ such that

$$f(x_*) = m, \quad f(x^*) = M.$$

Since $\mathrm{Mean}(f) \in [f(x_*), f(x^*)]$ we deduce from the Intermediate Value Theorem that there exists ξ in the interval $[x_*, x^*]$ such that $f(\xi) = \mathrm{Mean}(f)$. □

Theorem 9.42. *Suppose that $f : [a,b] \to \mathbb{R}$ is a Riemann integrable function. We define*

$$F : [a,b] \to \mathbb{R}, \quad F(x) := \int_a^x f(t)dt.$$

Then the following hold.

(i) *The function F is Lipschitz. In particular, F is continuous.*

(ii) *If the function f is continuous, then the function $F(x)$ is differentiable on $[a,b]$ and*

$$F'(x) = f(x), \quad \forall x \in [a,b].$$

In other words, $F(x)$ is an antiderivative of f, more precisely the unique antiderivative on $[a,b]$ such that $F(a) = 0$.

Proof. (i) We set

$$M := \sup_{x\in[a,b]} |f(x)|.$$

If $x, y \in [a,b]$, $x < y$, then

$$|F(x) - F(y)| = |F(y) - F(x)| = \left| \int_a^y f(t)dt - \int_a^x f(t)dt) \right|$$

$$= \left| \int_x^y f(t)dt \right| \leq \int_x^y |f(t)|dt \leq \int_x^y M dt = M(y-x) = M|x-y|.$$

This proves that F is Lipschitz.

(ii) We have to prove that if $x_0 \in [a,b]$, then

$$\lim_{x\to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

Using (9.27) we deduce

$$F(x) - F(x_0) = \int_a^x f(t)dt - \int_a^{x_0} f(t)dt = \int_{x_0}^x f(t)dt$$

so that we have to show that

$$\lim_{x\to x_0} \frac{1}{x-x_0} \int_{x_0}^x f(t)dt = f(x_0).$$

In other words, we have to prove that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\forall x \in [a,b], \;\; 0 < |x - x_0| < \delta \Rightarrow \left| \frac{1}{x-x_0} \int_{x_0}^x f(t)dt - f(x_0) \right| < \varepsilon. \tag{9.31}$$

Since f is continuous at x_0, given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\forall x \in [a,b], \;\; |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

On the other hand, invoking the continuity of f again, we deduce from the Integral Mean Value Theorem that, for any $x \neq x_0$, there exists ξ_x between x_0 and x such that

$$f(\xi_x) = \frac{1}{x-x_0} \int_{x_0}^x f(t)dt.$$

In particular, if $|x - x_0| < \delta$, then $|\xi_x - x_0| < \delta$, and thus

$$\left| \frac{1}{x-x_0} \int_{x_0}^x f(t)dt - f(x_0) \right| = |f(\xi_x) - f(x_0)| < \varepsilon.$$

□

9.6. How to compute a Riemann integral

To this day, the best method of computing by hand Riemann integrals is the fundamental theorem of calculus.

Theorem 9.43 (The Fundamental Theorem of Calculus: Part 1). *Suppose that $f : [a, b] \to \mathbb{R}$ is a function satisfying the following two conditions.*

(i) *The function f is Riemann integrable.*

(ii) *The function f admits antiderivatives on $[a, b]$.*

If $F : [a, b] \to \mathbb{R}$ is an antiderivative of f, then

$$\boxed{\int_a^b f(x)dx = F(x)\Big|_a^b := F(b) - F(a).} \tag{9.32}$$

More generally,

$$\boxed{\int_a^x f(t)dt = F(x) - F(a), \quad \forall x \in [a, b].} \tag{9.33}$$

Proof. Denote by $\boldsymbol{U}_n$ the uniform partition of $[a, b]$ of order n. Since f is Riemann integrable we deduce that *for any choices of samples* $\underline{\xi}^{(n)}$ of $\boldsymbol{U}_n$ we have

$$\int_a^b f(x)dx = \lim_{n\to\infty} \boldsymbol{S}\big(\, f, \boldsymbol{U}_n, \underline{\xi}^{(n)}\,\big).$$

The miracle is that for any n we can cleverly choose a sample

$$\underline{\xi}^{(n)} = (\xi_1^n, \dots, \xi_n^n)$$

of $\boldsymbol{U}_n$ such that the Riemann sum $\boldsymbol{S}\big(\, f, \boldsymbol{U}_n, \underline{\xi}^{(n)}\,\big)$ has an extremely simple form. Here are the details.

The k-th node of $\boldsymbol{U}_n$ is $x_k^n = a + \frac{k}{n}(b - a)$ and the k-th interval is $I_k = [x_{k-1}^n, x_k^n]$. The function F is differentiable on the closed interval $[a, b]$ and, in particular, it is continuous on $[a, b]$. We can invoke Lagrange's Mean Value Theorem to conclude that, for any $k = 1, \dots, n$, there exists $\xi_k^n \in (x_{k-1}^n, x_k^n)$ such that

$$f(\xi_k^n) = F'(\xi_k^n) = \frac{F(x_k^n) - F(x_{k-1}^n)}{x_k^n - x_{k-1}^n},$$

i.e.,

$$f(\xi_k^n)(x_k^n - x_{k-1}^n) = F(x_k^n) - F(x_{k-1}^n).$$

The collection $(\xi_1^n, \dots, \xi_n^n)$ is a sample $\underline{\xi}^{(n)}$ of the partition $\boldsymbol{U}_n$. The associated Riemann sum satisfies

$$\boldsymbol{S}\big(\, f, \boldsymbol{U}_n, \underline{\xi}^{(n)}\,\big) = f(\xi_1^n)(x_1^n - x_0^n) + f(\xi_2^n)(x_2^n - x_1^n) + \cdots + f(\xi_n^n)(x_n^n - x_{n-1}^n)$$

$$= F(x_1^n) - F(x_0^n) + F(x_2^n) - F(x_1^n) + \cdots + F(x_n^n) - F(x_{n-1}^n)$$

(the above is a telescopic sum!!!)

$$= F(x_n^n) - F(x_0^n) = F(b) - F(a).$$

Thus the sequence of Riemann sums $\boldsymbol{S}(f, \boldsymbol{U}_n, \underline{\xi}^{(n)})$ is constant, equal to $F(b) - F(a)$. Hence

$$\int_a^b f(x)dx = \lim_{n\to\infty} \boldsymbol{S}\big(f, \boldsymbol{U}_n, \underline{\xi}^{(n)}\big) = F(b) - F(a).$$

Applying (9.32) to the function The equality (9.33) follows by □

Corollary 9.44 (The Fundamental Theorem of Calculus: Part 2)**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a continuous function. Then f admits antiderivatives on $[a,b]$ and, if $F(x)$ is any antiderivative of f on $[a,b]$, then*

$$\int_a^b f(x)dx = F\Big|_a^b := F(b) - F(a), \;\; F(x) = F(a) + \int_a^x f(t)dt, \;\; \forall x \in [a,b]. \tag{9.34}$$

Proof. The fact that f admits antiderivatives follows from Theorem 9.42(b). The rest follows from Theorem 9.43. □

Remark 9.45. (a) Theorem 9.43 shows that the computation of Riemann integral of a function can be reduced to the computation of the antiderivatives of that function, if they exist. As we have seen in the previous chapter, for many classes of continuous function this computation can be carried out successfully in a *finite number* of purely algebraic steps.

If we ponder for a little bit, the equality (9.32) is a truly remarkable result. The left-had-side of (9.32) is a Riemann integral defined by a very laborious limiting process which involves *infinitely many and computationally very punishing steps*. The right-hand-side of (9.32) involves computing the values of an antiderivative at two points. Often this can be achieved in *finitely many arithmetic steps*!

The attribute *fundamental* attached to Theorem 9.43 is fully justified: it describes a *finite-time* shortcut to an *infinite-time* process.

(b) Both assumptions (i) and (ii) are needed in Theorem 9.43! Indeed, there exist functions that satisfy (i) but not (ii), and there exist function satisfying (ii), but not (i). Their constructions are rather ingenious and we refer to [**5**] for more details. Note that the continuous functions automatically satisfy both (i) and (ii). □

Example 9.46. For $k \in \mathbb{N}$ consider the continuous function $f : [0,1] \to \mathbb{R}$, $f(x) = x^k$. The function $F(x) = \frac{1}{k+1}x^{k+1}$ is an antiderivative of f and (9.34) implies

$$\int_0^1 x^k dx = \Big(\frac{1}{k+1}x^{k+1}\Big)\Big|_0^1 = \frac{1}{k+1}.$$

In particular, for $k = 2$ we deduce

$$\int_0^1 x^2 dx = \frac{1}{3}.$$

This agrees with the elementary computations in Section 9.1. □

The techniques for computing antiderivatives can now be used for computing Riemann integrals. As we have seen, there are basically two methods for computing antiderivatives: integration by parts, and change of variables. These lead to two basic techniques for computing

Riemann integrals. In applications most often one needs to use a blend of these techniques to compute a Riemann integral.

9.6.1. Integration by parts. We state a special case that covers most of the concrete situations.

Proposition 9.47. *Suppose that that $u, v : [a, b] \to \mathbb{R}$ are two C^1 functions, i.e., they are differentiable have continuous derivatives. Then uv' and $u'v$ are Riemann integrable and*

$$\boxed{\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)dx}. \tag{9.35}$$

Proof. The functions $u'v$ and uv' are continuous since they are products of continuous functions. In particular these functions are integrable, and we have

$$\begin{aligned}\int_a^b u'(x)v(x)dx + \int_a^b u(x)v'(x)dx &= \int_a^b \big(\, u'(x)v(x) + u(x)v'(x)\,\big)dx \\ &= \int_a^b (uv)'(x)dx \stackrel{(9.32)}{=} u(x)v(x)\Big|_a^b.\end{aligned}$$

The equality (9.35) is now obvious. □

Remark 9.48. The integration-by-parts formula (9.35) is often written in the shorter form

$$\boxed{\int_a^b udv = uv\Big|_a^b - \int_a^b vdu.} \tag{9.36}$$

Observing that

$$uv\Big|_b^a = u(a)v(a) - u(b)v(b) = -\big(\, u(b)v(b) - u(a)v(a)\,\big) = -uv\Big|_a^b,$$

we deduce that

$$\int_b^a udv = uv\Big|_b^a - \int_b^a vdu,$$

even though the upper limit of integration a is smaller than the lower limit of integration b.□

Example 9.49. For any nonnegative integers m, n we set

$$I_{m,n} = \int_{-1}^1 (x-1)^m(x+1)^n dx. \tag{9.37}$$

This integral is theoretically computable because $(x-1)^m(x+1)^n$ is a polynomial. Its precise form is obtained via Newton's binomial formula and the final result is rather complicated. For example

$$(x-1)^2(x+1)^3 = (x^2-2x+1)(x^3+3x^2+3x+1) = x^5 + x^4 - 2\,x^3 - 2\,x^2 + x + 1.$$

In general, we need to multiply the two polynomials in the right-hand-side of (9.37) to obtain the explicit form of $(x-1)^m(x+1)^n$. This is an elaborate process which becomes increasingly more complex as the powers m and n increase. However, an ingenious usage of the integration-by-parts trick leads to a much simpler way of computing $I_{m,n}$.

Let us first observe that

$$(x+1)^n = \frac{1}{n+1}\frac{d}{dx}(x+1)^{n+1},$$

from which we deduce

$$I_{0,n} = \int_{-1}^{1}(x+1)^n dx = \frac{1}{n+1}(x+1)^{n+1}\Big|_{-1}^{1} = \frac{2^{n+1}}{n+1}. \tag{9.38}$$

Observe now that if $m > 0$, then

$$\begin{aligned} I_{m,n} &= \int_{-1}^{1}(x-1)^m(x+1)^n dx = \frac{1}{n+1}\int_{-1}^{1}(x-1)^m\frac{d}{dx}(x+1)^{n+1}dx \\ &= \underbrace{\frac{1}{n+1}(x-1)^m(x+1)^{n+1}\Big|_{-1}^{1}}_{=0} - \frac{m}{n+1}\int_{-1}^{1}(x-1)^{m-1}(x+1)^{n+1}dx. \end{aligned}$$

We obtain in this fashion the recurrence relation

$$I_{m,n} = -\frac{m}{n+1}I_{m-1,n+1}, \quad \forall m > 0, \quad n \geq 0. \tag{9.39}$$

If $m-1 > 0$, then we can continue this process and we deduce

$$I_{m-1,n+1} = -\frac{m-1}{n+2}I_{m-2,n+2} \Rightarrow I_{m,n} = \frac{m(m-1)}{(n+1)(n+2)}I_{m-2,n+2}.$$

Iterating this procedure we conclude that

$$\begin{aligned} I_{m,n} &= (-1)^m\frac{m(m-1)\cdots 2\cdot 1}{(n+1)(n+2)\cdots(n+m-1)(n+m)}I_{0,n+m} \\ &= (-1)^m\frac{m!}{(n+1)\cdots(n+m)}I_{0,n+m} = (-1)^m\frac{1}{\binom{n+m}{m}}I_{0,n+m}. \end{aligned}$$

Invoking (9.38) we deduce

$$I_{m,n} = (-1)^m\frac{1}{\binom{n+m}{m}}\cdot\frac{2^{n+m+1}}{(n+m+1)}. \tag{9.40}$$

When $m = n$ we have

$$I_{n,n} = \int_{-1}^{1}(x-1)^n(x+1)^n dx = \int_{-1}^{1}(x^2-1)^n dx$$

and we conclude that

$$\int_{-1}^{1}(x^2-1)^n dx = I_{n,n} = \frac{(-1)^n}{\binom{2n}{n}}\cdot\frac{2^{2n+1}}{(2n+1)}. \tag{9.41}$$

□

Example 9.50 (Wallis' formula)**.** For nonnegative integer n we set

$$I_n := \int_0^{\frac{\pi}{2}}(\sin x)^n\, dx.$$

Note that

$$I_0 = \frac{\pi}{2}, \quad I_1 = \int_0^{\frac{\pi}{2}}\sin x\, dx = (-\cos x)\Big|_{x=0}^{x=\frac{\pi}{2}} = 1.$$

In general, we have

$$I_{n+1} = \int_0^{\frac{\pi}{2}} (\sin x)^n d(-\cos x) = (\sin x)^n(-\cos x)\Big|_{x=0}^{x=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, d(\sin x)^n$$

$$= n\int_0^{\frac{\pi}{2}} (\sin x)^{n-1}\cos^2 x \, dx = n\int_0^{\frac{\pi}{2}} (\sin x)^{n-1}(1-\sin^2 x)\, dx = nI_{n-1} - nI_{n+1}.$$

Hence

$$(n+1)I_{n+1} = nI_{n-1}, \;\; I_{n+1} = \frac{n}{n+1}I_{n-1}. \tag{9.42}$$

We deduce

$$I_2 = \frac{1}{2}I_0 = \frac{1}{2}\frac{\pi}{2}, \;\; I_4 = \frac{3}{4}I_2 = \frac{3}{4}\frac{1}{2}\frac{\pi}{2},$$

and, in general,

$$I_{2n} = \frac{2n-1}{2n}\cdots\frac{3}{4}\frac{1}{2}\frac{\pi}{2}. \tag{9.43}$$

Similarly,

$$I_3 = \frac{2}{3}I_1 = \frac{2}{3}, \;\; I_5 = \frac{4}{5}I_3 = \frac{4}{5}\frac{2}{3},$$

and, in general,

$$I_{2n+1} = \frac{2n}{2n+1}\cdots\frac{4}{5}\frac{2}{3}. \tag{9.44}$$

Since $\sin x \in [0,1]$, $\forall x \in [0,\pi/2]$ we deduce

$$(\sin x)^{n+1} \leq (\sin x)^n, \;\; \forall x \in [0,\pi/2],$$

and thus,

$$I_{n+1} \leq I_n, \;\; \forall n \in \mathbb{N}.$$

We deduce

$$\frac{2n}{2n+1} \overset{(9.42)}{=} \frac{I_{2n+1}}{I_{2n-1}} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1.$$

From the above equalities we deduce

$$\lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = 1.$$

Using (9.43) and (9.44) we deduce

$$\frac{I_{2n+1}}{I_{2n}} = \frac{2}{\pi}\cdot\frac{1}{2n+1}\cdot\frac{2^2 4^2\cdots(2n)^2}{1^2 3^2\cdots(2n-1)^2}.$$

This implies the celebrated *Wallis' formula*

$$\boxed{\frac{\pi}{2} = \frac{\pi}{2}\lim_{n\to\infty}\frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty}\frac{2^2 4^2\cdots(2n)^2}{1^2 3^2\cdots(2n-1)^2}\cdot\frac{1}{2n+1}.} \tag{9.45}$$

Later on, we will need an equivalent version of the above equality

$$\boxed{\frac{\pi}{2} = \frac{\pi}{2}\lim_{n\to\infty}\frac{2n+1}{2n}\frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty}\frac{2^2 4^2\cdots(2n)^2}{1^2 3^2\cdots(2n-1)^2}\cdot\frac{1}{2n}.} \tag{9.46}$$

□

Let us discuss another simple but useful application of the integration-by-parts trick.

Proposition 9.51 (Integral remainder formula)**.** *Let $n \in \mathbb{N}$ and suppose that $f:[a,b]\to\mathbb{R}$ is a C^{n+1}-function, i.e., $(n+1)$-times differentiable and the $(n+1)$-th derivative is continuous. If $x_0\in[a,b]$ and $T_n(x)$ is the degree n-Taylor polynomial of f at x_0,*

$$T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

then the remainder $R_n(x) := f(x) - T_n(x)$ admits the integral representation

$$\boxed{R_n(x) = \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^n dt, \quad \forall x\in[a,b]}. \tag{9.47}$$

Proof. Fix $x \neq x_0$. We have

$$\begin{aligned}
f(x)-f(x_0) &= \int_{x_0}^x f'(t)dt = -\int_{x_0}^x f'(t)\frac{d}{dt}(x-t)\,dt\\
&= -\Big(f'(t)(x-t)\Big)\Big|_{t=x_0}^{t=x} + \int_{x_0}^x f''(t)(x-t)dt\\
&= f'(x_0)(x-x_0) - \int_{x_0}^x f''(t)\frac{d}{dt}\left(\frac{1}{2}(x-t)^2\right)dt\\
&= f'(x_0)(x-x_0) - \left(\frac{1}{2}f''(t)(x-t)^2\right)\Big|_{t=x_0}^{t=x} + \frac{1}{2}\int_{x_0}^x f^{(3)}(t)(x-t)^2dt\\
&= f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 - \frac{1}{3!}\int_{x_0}^x f^{(3)}(t)\frac{d}{dt}(x-t)^3dt\\
&= f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 - \frac{1}{3!}\big(f^{(3)}(t)(x-t)^3\big)\Big|_{t=x_0}^{t=x} + \frac{1}{3!}\int_{x_0}^x f^{(4)}(t)(x-t)^3dt\\
&= f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \frac{1}{3!}\int_{x_0}^x f^{(4)}(t)(x-t)^3dt\\
&= f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 - \frac{1}{4!}\int_{x_0}^x f^{(4)}(t)\frac{d}{dt}(x-t)^4dt\\
&= \cdots\cdots\cdots\cdots\cdots =\\
&= f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^ndt.
\end{aligned}$$

Thus

$$\begin{aligned}
f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\\
&\quad + \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^ndt\\
&= T_n(x) + \frac{1}{n!}\int_{x_0}^x f^{(n+1)}(t)(x-t)^ndt.
\end{aligned}$$

This proves (9.47). □

Example 9.52. Let us show how we can use the integral remainder formula to strengthen the result in Exercise 8.7. Consider the function $f : (-1,1) \to \mathbb{R}$, $f(x) = \ln(1-x)$. Since

$$f'(x) = -\frac{1}{1-x} = (x-1)^{-1}, \;\; f''(x) = \frac{d}{dx}(x-1)^{-1} = -(x-1)^{-2},$$

$$f^{(3)}(x) = -\frac{d}{dx}(x-1)^{-2} = 2(x-1)^{-3}, \ldots$$

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(x-1)^{-n}, \;\; \forall n \in \mathbb{N}$$

we deduce that

$$f(0) = 0, \;\; f^{(n)}(0) = (-1)^n(n-1)!(-1)^{-n} = -(n-1)!, \;\; \forall n \in \mathbb{N},$$

and thus, the Taylor series of f at $x_0 = 0$ is

$$-\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

We denote by $T_n(x)$ the degree n Taylor polynomial of $f(x)$ at $x_0 = 0$,

$$T_n(x) = -\sum_{k=1}^{n} \frac{x^k}{k} = -x - \frac{x^2}{2} - \cdots - \frac{x^n}{n!}.$$

We want to prove that this series converges to $\ln(1-x)$ for any $x \in [-1,1)$. To do this we have to show that

$$\lim_{n\to\infty} |f(x) - T_n(x)| = 0, \;\; \forall x \in [-1,1).$$

We need to estimate the remainder $R_n(x) = f(x) - T_n(x)$. We distinguish two cases.

1. $x \in [0,1)$. Using the integral remainder formula (9.47) we deduce

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n dt = (-1)^n \int_0^x (t-1)^{-n-1}(x-t)^n dt.$$

Hence

$$|R_n(x)| = \int_0^x \frac{(x-t)^n}{(1-t)^{n+1}} dt.$$

Observe that for $t \in [0,x]$ we have $1 - t \geq 1 - x > 0$ so that, for any $t \in [0,x]$ we have

$$(1-t)^{n+1} \geq (1-t)^n(1-x) > 0, \Longleftrightarrow 0 < \frac{1}{(1-t)^{n+1}} \leq \frac{1}{1-x} \cdot \frac{1}{(1-t)^n}.$$

Hence

$$|R_n(x)| \leq \frac{1}{1-x}\int_0^x \left(\frac{x-t}{1-t}\right)^n dt.$$

Now consider the function

$$g : [0,x] \to \mathbb{R}, \;\; g(t) = \frac{x-t}{1-t}.$$

We have

$$g'(t) = \frac{-(1-t) + (x-t)}{(1-t)^2} = \frac{x-1}{(1-t)^2} < 0.$$

Hence

$$0 = g(x) \leq g(t) \leq g(0) = x, \;\; \forall t \in [0,x],$$

and thus

$$|R_n(x)| \leq \frac{1}{1-x}\int_0^x g(t)^n dt \leq \frac{1}{1-x}\int_0^x x^n dt = \frac{x^{n+1}}{1-x}.$$

We deduce

$$|R_n(x)| \leq \frac{x^{n+1}}{1-x}, \ \ \forall x \in [0,1),$$

so that

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{x^{n+1}}{1-x} = 0, \ \ \forall x \in [0,1).$$

2. $x \in [-1,0)$. We estimate $R_n(x)$ using the Lagrange remainder formula. Hence, there exists $\xi \in (x,0)$ such that

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1} = (-1)^n \frac{n!(\xi-1)^{-(n+1)}}{(n+1)!}x^{n+1} = (-1)^n \frac{1}{(n+1)(\xi-1)^{n+1}}x^{n+1}.$$

Hence, since $\xi \in (x,0)$, we have $|\xi - 1| = |\xi| + 1$ and

$$|R_n(x)| = \frac{|x|^{n+1}}{(n+1)(1+|\xi|)^{n+1}} \leq \frac{|x|^{n+1}}{n+1}.$$

Since $|x| \leq 1$ we deduce

$$\lim_{n\to\infty} |R_n(x)| = 0, \ \ \forall x \in [-1,0).$$

We have thus proved that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \ \ \forall x \in [-1,1).$$

Note in particular that

$$f(-1) = \ln 2 = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots . \tag{9.48}$$

□

9.6.2. Change of variables. The change of variables in the Riemann integral is very similar to the integration-by-substitution trick used in the computation of antiderivatives, but it has a few peculiarities. There are two versions of the change of variables formula.

Proposition 9.53 (Change of variables formula: version 1)**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is a continuous function and $\varphi : [\alpha,\beta] \to [a,b]$ is a C^1-function. Then the function $f\big(\varphi(t)\big)\varphi'(t)$ is integrable on $[\alpha,\beta]$ and*

$$\boxed{\int_{\alpha}^{\beta} f\big(\varphi(t)\big)\varphi'(t)dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.} \tag{9.49}$$

Proof. Since f is continuous it admits antiderivatives. Fix an antiderivative F of f. The chain rule shows that $F\big(\varphi(t)\big)$ is an antiderivative of the continuous function $f\big(\varphi(t)\big)\varphi'(t)$. The Fundamental Theorem of Calculus then shows

$$\int_{\alpha}^{\beta} f\big(\varphi(t)\big)\varphi'(t)dt = F\big(\varphi(t)\big)\Big|_{t=\alpha}^{t=\beta} = F\big(\varphi(\beta)\big) - F\big(\varphi(\alpha)\big) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx.$$

□

We can relax the continuity assumption of f, but to do so we need to make an additional assumption of φ.

Proposition 9.54 (Change of variables formula: version 2)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function and $\varphi : [\alpha, \beta] \to [a, b]$ is a C^1-function such that*

$$\varphi'(t) \neq 0, \quad \forall t \in (\alpha, \beta).$$

Then $f\big(\varphi(t)\big)\varphi'(t)$ is Riemann integrable on $[\alpha, \beta]$ and

$$\boxed{\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f\big(\varphi(t)\big)\varphi'(t)dt.} \tag{9.50}$$

Proof. Set

$$M := \sup_{t\in[\alpha,\beta]} |\varphi'(t)|.$$

Note that $M > 0$. Since $|\varphi'(t)|$ is continuous, Weierstrass' theorem implies that $M < \infty$. Since $\varphi'(t) \neq 0$ for any $t \in (\alpha, \beta)$ we deduce from the Intermediate Value Theorem that

$$\text{either } \varphi'(t) > 0, \ \forall t \in (\alpha, \beta) \text{ or; } \varphi'(t) < 0, \ \forall t \in (\alpha, \beta).$$

Thus, either φ is strictly increasing and its range is $[\varphi(\alpha), \varphi(\beta)]$, or φ is strictly decreasing and its range is $[\varphi(\beta), \varphi(\alpha)]$. We need to discuss each case separately, but we will present the details only for the first case and leave the details for the second case for you as an exercise. In the sequel we will assume that φ is increasing and thus

$$0 < \varphi'(t) \leq M, \ \forall t \in (\alpha, \beta).$$

For simplicity we set

$$g(t) := f\big(\varphi(t)\big)\varphi'(t), \ \ t \in [\alpha, \beta].$$

We will show that g is Riemann integrable on $[\alpha, \beta]$ and its Riemann integral is given by the left-hand side of (9.50). We will we will need the following technical result.

Lemma 9.55. *For any partition $\boldsymbol{P}$ of $[\alpha, \beta]$, there exists a partition $\boldsymbol{P}_\varphi$ of $[\varphi(\alpha), \varphi(\beta)]$ and samples $\underline{\xi}$ of $\boldsymbol{P}$ and $\underline{\eta}$ of $\boldsymbol{P}_\varphi$ such that*

$$\|\boldsymbol{P}_\varphi\| \leq M\|\boldsymbol{P}\|, \tag{9.51a}$$

$$\boldsymbol{S}(\boldsymbol{P}, g, \underline{\xi}) = \boldsymbol{S}(\boldsymbol{P}_\varphi, f, \underline{\xi}_\varphi). \tag{9.51b}$$

Let us first show that Lemma 9.55 implies that g is Riemann integrable and satisfies (9.50). Fix $\varepsilon > 0$. The function f is Riemann integrable on $[\varphi(\alpha), \varphi(\beta)]$ and thus there exists $\delta_0 = \delta_0(\varepsilon) > 0$ such that for any partition $\boldsymbol{Q}$ of $[\varphi(\alpha), \varphi(\beta)]$ and any sample $\underline{\eta}$ of $\boldsymbol{Q}$ we have

$$\left| \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx - \boldsymbol{S}(f, \boldsymbol{Q}, \underline{\eta}) \right| < \varepsilon. \tag{9.52}$$

Set

$$\delta = \delta(\varepsilon) := \frac{1}{M}\delta_0(\varepsilon).$$

For any partition $\boldsymbol{P}$ of $[\alpha, \beta]$ of mesh $\|\boldsymbol{P}\| < \delta(\varepsilon)$, and any sample $\underline{\xi}$ of $\boldsymbol{P}$, the sampled partition $(\boldsymbol{P}_\varphi, \underline{\xi}_\varphi)$ of $[\varphi(\alpha), \varphi(\beta)]$ associated to $(\boldsymbol{P}, \underline{\xi})$ by Lemma 9.55 satisfies

$$\boldsymbol{P}_\varphi\| < M\delta(\varepsilon) = \delta_0(\varepsilon) \ \text{ and } \ \boldsymbol{S}(g, \boldsymbol{P}, \underline{\xi}) = \boldsymbol{S}(f, \boldsymbol{P}_\varphi, \underline{\xi}_\varphi).$$

We deduce that

$$\left| \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx - \boldsymbol{S}(g, \boldsymbol{P}, \underline{\xi}) \right| = \left| \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx - \boldsymbol{S}(f, \boldsymbol{P}_\varphi, \underline{\xi}_\varphi) \right| \stackrel{(9.52)}{<} \varepsilon.$$

This proves that $g(t)$ is integrable on $[\alpha, \beta]$ and its integral is equal to $\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx$.

Proof of Lemma 9.55. Consider a partition $\boldsymbol{P} = (\alpha = t_0 < t_1 < \cdots < t_n = \beta)$ of $[\alpha, \beta]$. For $k = 0, 1, \ldots, x_n$ we set

$$x_k := \varphi(t_k).$$

Since φ is increasing we have

$$x_{k-1} < x_k,\ \ \forall k = 1, \dots, n.$$

Thus

$$\varphi(\alpha) = x_0 < x_1 < \cdots < x_n = \varphi(\beta)$$

is a partition of $[\varphi(\alpha), \varphi(\beta)]$ that we denote by $\boldsymbol{P}_\varphi$. Note that

$$x_k - x_{k-1} = \varphi(t_k) - \varphi(t_{k-1}).$$

Lagrange's Mean Value theorem implies that there exists $\xi_k \in (t_{k-1}, t_k)$ such that

$$x_k - x_{k-1} = \varphi(t_k) - \varphi(t_{k-1}) = \varphi'(\xi_k)(t_k - t_{k-1}).$$

In particular, this shows that

$$|x_{k-1} - x_k| = |\varphi'(\xi_k)| \cdot |t_k - t_{k-1}| \leq M|t_k - t_{k-1}|,\ \ \forall k = 1, \dots, k.$$

Hence

$$\|\boldsymbol{P}_\varphi\| \leq M\|\boldsymbol{P}\|.$$

This proves (9.51a).

Set $\eta_k := \varphi(\xi_k)$. Note that since φ is increasing we have $\eta_k \in (x_{k-1}, x_k)$. The collection $\underline{\xi} = (\xi_1, \dots, \xi_k)$ is a sample of $\boldsymbol{P}$, and the collection $\underline{\eta} = (\eta_1, \dots, \eta_n)$ is a sample of $\boldsymbol{P}_\varphi$. Observe that

$$f(\eta_k)(x_k - x_{k-1}) = f\big(\varphi(\xi_k)\big)\varphi'(\xi_k)(t_k - t_{k-1}) = g(\xi_k)(t_k - t_{k-1}).$$

Thus

$$\boldsymbol{S}(f, \boldsymbol{P}_\varphi, \underline{\eta}) = \sum_{k=1}^n f(\eta_k)(x_k - x_{k-1}) = \sum_{k=1}^n g(\xi_k)(t_k - t_{k-1}) = \boldsymbol{S}(g, \boldsymbol{P}, \underline{\xi}).$$

This proves (9.51b) and completes the proof of Proposition 9.54. □

Remark 9.56. In concrete examples, the right-hand sides of the equalities (9.49) and (9.50) are quantities that we know how to compute. The left-hand sides are the unknown quantities whose computations are sought. For this reason these two equalities play different roles in applications. □

Example 9.57. (a) Suppose that we want to compute

$$\int_{-1}^2 \cos(t^2)tdt = \frac{1}{2}\int_{-1}^2 \cos(t^2)d(t^2).$$

We make the change of variables $x = t^2$. Note that $t = -1 \Rightarrow x = 1$, $t = 2 \Rightarrow x = 4$ and we deduce

$$\int_{-1}^2 \cos(t^2)tdt \stackrel{(9.49)}{=} \frac{1}{2}\int_1^4 \cos x\, dx = \frac{\sin 4 - \sin 1}{2}.$$

Note that in this case (9.50) is not applicable.

(b) Suppose that we want to compute

$$\int_0^{\frac{\pi}{2}} e^{\sin t}\cos t\, dt = \int_0^{\frac{\pi}{2}} e^{\sin t} d(\sin t).$$

We make the change in variables $x = \sin t$. Note that $t = 0 \Rightarrow x = 0$, $t = \frac{\pi}{2} \Rightarrow x = 1$ and we deduce

$$\int_0^{\frac{\pi}{2}} e^{\sin t}\cos tdt \stackrel{(9.49)}{=} \int_0^1 e^x dx = e^x\Big|_{x=0}^{x=1} = e - 1.$$

(c) Suppose we want to compute

$$\int_{-1}^1 \sqrt{1 - x^2}dx$$

We make a change of variables $x = \sin t$ so that $dx = d(\sin t) = \cos t dt$. Note that

$$x = -1 \Rightarrow t = -\frac{\pi}{2}, \;\; x = 1 \Rightarrow t = \frac{\pi}{2},$$

and $\cos t > 0$ when $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Hence

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t, \;\; -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

We deduce

$$\int_{-1}^{1} \sqrt{1-x^2} dx \stackrel{(9.50)}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} dt$$

$$= \frac{1}{2}\Big(\frac{\pi}{2} + \frac{\pi}{2}\Big) + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2t dt = \frac{\pi}{2} + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2t dt.$$

To compute the last integral we use the change invariables $u = 2t$ so that $dt = \frac{1}{2}du$,

$$t = -\frac{\pi}{2} \Rightarrow u = \pi, \;\; t = \frac{\pi}{2} \Rightarrow u = \pi.$$

Hence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2t \, dt = \frac{1}{2}\int_{-\pi}^{\pi} \cos u \, du = \frac{1}{2}\big(\sin \pi - \sin(-\pi)\big) = 0.$$

We conclude that

$$\int_{-1}^{1} \sqrt{1-x^2} dx = \frac{\pi}{2}. \tag{9.53}$$

Let us observe that this equality provides a way of approximating $\frac{\pi}{2}$ by using Riemann sums to approximate the integral in the left-hand side. If we use the uniform partition $\boldsymbol{U}_{200}$ of order 200 of $[-1, 1]$and as sample $\underline{\xi}$ the right end points of the intervals of the partition, then we deduce

$$\pi \approx 2\boldsymbol{S}\big(\sqrt{1-x^2}, \boldsymbol{U}_{200}, \underline{\xi}\big) \approx 3.14041.....$$

If we use the uniform partition of order $2,000$ and a similar sample, then we deduce

$$\pi \approx 2\boldsymbol{S}\big(\sqrt{1-x^2}, \boldsymbol{U}_{2,000}, \underline{\xi}\big) \approx 3.14157.....$$

(d) Suppose that we want to compute the integral.

$$\int_{1}^{e} \frac{\ln x}{x} dx.$$

We make the change in variables $x = e^t$ and we observe that $dx = e^t dt$,

$$x = 1 \Rightarrow t = 0, \;\; x = e \Rightarrow t = 1.$$

The derivative $\frac{dx}{dt} = e^t$ is everywhere positive and we deduce

$$\int_{1}^{e} \frac{\ln x}{x} dx \stackrel{(9.50)}{=} \int_{0}^{1} \frac{\ln e^t}{e^t} e^t dt = \int_{0}^{1} t dt = \frac{t^2}{2}\Big|_{t=0}^{t=1} = \frac{1}{2}. \qquad \square$$

Example 9.58 (Stirling's formula)**.** We want to estimate the size of

$$F_n := \ln(1) + \ln(2) + \cdots + \ln n = \ln(2) + \cdots + \ln n = \ln(1 \cdot 2 \cdots n) = \ln(n!).$$

Observe that

$$\int_{k-1/2}^{k+1/2} \ln t dt = \int_{k}^{k+1/2} \ln t dt + \int_{k-1/2}^{k} \ln t dt$$

(use the change of variables $t = k + s$ in the first integral and $t = k - s$ in the second)

$$= \int_0^{\frac{1}{2}} \ln(k+s)ds + \int_0^{\frac{1}{2}} \ln(k-s)ds = \int_0^{\frac{1}{2}} \Big(\ln(k+s) + \ln(k-s) \Big) ds$$

$$= \int_0^{\frac{1}{2}} \ln(k^2 - s^2)ds = \int_0^{\frac{1}{2}} \left(\ln k^2 + \ln\Big(1 - \frac{s^2}{k^2}\Big) \right) ds$$

$(\ln k^2 = 2\ln k)$

$$= \ln k + \int_0^{\frac{1}{2}} \ln\Big(1 - \frac{s^2}{k^2}\Big) ds.$$

Hence

$$\ln k = \int_{k-1/2}^{k+1/2} \ln t dt \underbrace{- \int_0^{\frac{1}{2}} \ln\Big(1 - \frac{s^2}{k^2}\Big) ds}_{=: r_k} = \int_{k-1/2}^{k+1/2} \ln t\, dt + r_k. \tag{9.54}$$

Consider the function

$$f : [0,1) \to \mathbb{R}, \quad f(x) = \ln(1-x).$$

Note that $f(x) \leq 0$, $\forall x \in [0,1)$. Moreover, $f(x)$ is convex so its graph sits above its linearization at $x_0 = 0$ which is $L(x) = -x$. Thus

$$-x \leq \ln(1-x) \leq 0, \quad \forall x \in [0,1).$$

Thus

$$-\frac{1}{24k^2} = -\int_0^{\frac{1}{2}} \frac{s^2}{k^2} ds \leq \int_0^{\frac{1}{2}} \ln\Big(1 - \frac{s^2}{k^2}\Big) ds =: -r_k < 0$$

so that

$$0 < r_k \leq \frac{1}{24k^2}, \quad \forall k \in \mathbb{N}. \tag{9.55}$$

In particular, this shows that the series $\sum_{k \geq 1} r_k$ is convergent and we denote by R is sum,

$$R := \sum_{k \geq 1} r_k.$$

Summing (9.54) from $k = 1$ to n we deduce

$$F_n = \sum_{k=1}^n \ln k = \sum_{k=1}^n \int_{k-1/2}^{k+1/2} \ln t dt + \sum_{k=1}^n r_k = \int_{\frac{1}{2}}^{n+\frac{1}{2}} \ln t dt + \underbrace{\sum_{k=1}^n r_k}_{=: R_n}. \tag{9.56}$$

We've shown in Example 8.46(a) that $x \ln x - x$ is an antiderivative of $\ln x$ so that

$$\int_{\frac{1}{2}}^{n+\frac{1}{2}} \ln t\, dt = \left(n + \frac{1}{2}\right) \ln\left(n + \frac{1}{2}\right) - \left(n + \frac{1}{2}\right) - \frac{1}{2} \ln\frac{1}{2} + \frac{1}{2}$$

$$= \left(n + \frac{1}{2}\right) \ln\left(n + \frac{1}{2}\right) - n + \frac{\ln 2}{2}$$

$$= \left(n + \frac{1}{2}\right) \ln n + \left(n + \frac{1}{2}\right) \left(\ln\left(n + \frac{1}{2}\right) - \ln n \right) + \frac{\ln 2}{2}$$

$$= \left(n + \frac{1}{2}\right) \ln n - n + \underbrace{\left(n + \frac{1}{2}\right) \ln\left(\frac{2n+1}{2n}\right) + \frac{\ln 2}{2}}_{=: Z_n}.$$

Let notice that

$$\left(n+\frac{1}{2}\right)\ln\left(\frac{2n+1}{2n}\right)=\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{2n}\right)$$
$$=\frac{n+\frac{1}{2}}{2n}\cdot\frac{\ln\left(1+\frac{1}{2n}\right)}{\frac{1}{2n}}\to\frac{1}{2}\ \text{ as } n\to\infty.$$

This shows that

$$\lim_{n\to\infty} Z_n = Z := \frac{1+\ln 2}{2}.$$

Using (9.56) we deduce that

$$F_n=\left(n+\frac{1}{2}\right)\ln n - n + Z_n + R_n.$$

In particular

$$n! = e^{F_n} = e^{\left(n+\frac{1}{2}\right)\ln n - n + Z_n + R_n} = e^{n\ln n - n}e^{\ln\sqrt{n}}e^{Z_n+R_n} = \sqrt{n}\left(\frac{n}{e}\right)^n e^{Z_n+R_n},$$

and we deduce,

$$\lim_{n\to\infty}\frac{n!}{\sqrt{n}\left(\frac{n}{e}\right)^n} = \lim_{n\to\infty} e^{Z_n+R_n} = e^{Z+R}. \tag{9.57}$$

We set

$$C := e^{Z+R},\ \ S_n := \sqrt{n}\left(\frac{n}{e}\right)^n,$$

so that

$$\lim_{n\to\infty}\frac{n!}{CS_n}=1. \tag{9.58}$$

Thus, as $n\to\infty$, the factorial $n!$ behaves like CS_n. We rewrite the above equality as

$$n! \sim CS_n = C\sqrt{n}\left(\frac{n}{e}\right)^n.$$

To find the mysterious constant C we rely on Wallis' formula (9.46) which states that

$$\frac{\pi}{2}=\lim_{n\to\infty}\frac{2^2 4^2\cdots(2n)^2}{1^2 3^2\cdots(2n-1)^2}\cdot\frac{1}{2n}.$$

Now observe that

$$\frac{2^2 4^2\cdots(2n)^2}{1^2 3^2\cdots(2n-1)^2}\cdot\frac{1}{2n}=\frac{(n!)^2 2^{2n}}{1^2 3^2\cdots(2n-1)^2}\cdot\frac{1}{2n}=\frac{(n!)^4 2^{4n}}{(2n!)^2(2n)}.$$

Hence

$$\sqrt{\frac{\pi}{2}}=\lim_{n\to\infty}\frac{(n!)^2 2^{2n}}{(2n!)\sqrt{2n}},$$

i.e.,

$$\sqrt{\pi}=\lim_{n\to\infty}\frac{(n!)^2 2^{2n}}{(2n!)\sqrt{n}}=\lim_{n\to\infty}\frac{C^2S_n^2 2^{2n}}{CS_{2n}\sqrt{n}}\cdot\frac{\left(\frac{n!}{CS_n}\right)^2 2^{2n}}{\frac{(2n)!}{CS_{2n}}}\overset{(9.58)}{=}\lim_{n\to\infty}\frac{C^2S_n^2 2^{2n}}{CS_{2n}\sqrt{n}}=C\lim_{n\to\infty}\frac{S_n^2 2^{2n}}{S_{2n}\sqrt{n}}.$$

Now observe that

$$S_n=\sqrt{n}\left(\frac{n}{e}\right)^n \Rightarrow S_n^2=\frac{n^{2n+1}}{e^{2n}},\ \ S_{2n}=\sqrt{2n}\frac{(2n)^{2n}}{e^{2n}}=2^{2n}\sqrt{2n}\frac{n^{2n}}{e^{2n}},$$

and thus

$$\frac{S_n^2 2^{2n}}{S_{2n}\sqrt{n}} = \frac{2^{2n}\frac{n^{2n+1}}{e^{2n}}}{2^{2n}\sqrt{2n}\cdot\frac{n^{2n}}{e^{2n}}\cdot\sqrt{n}} = \frac{1}{\sqrt{2}}.$$

Hence

$$\sqrt{\pi} = C\lim_{n\to\infty}\frac{S_{2n}\sqrt{n}}{2^{2n}S_n^2} = \frac{C}{\sqrt{2}} \Rightarrow C = \sqrt{2\pi}.$$

We have thus proved *Stirling's formula*

$$\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n\to\infty}, \tag{9.59}$$

i.e.,

$$\boxed{\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1.}$$

□

9.7. Improper integrals

The Riemann integral is an operation defined for certain *bounded* functions defined on *bounded* intervals. Sometimes, even when one or both of these boundedness requirements is violated we can still give a meaning to an integral. Before we proceed with rigorous definitions it is helpful to look at some guiding examples.

Example 9.59. (a) Let $\alpha\in(0,1)$ and consider the function

$$f:(0,1]\to\mathbb{R},\quad f(x)=\frac{1}{x^\alpha}.$$

This function is continuous on $(0,1]$, but it is not bounded on this interval because

$$\lim_{x\to 0+}\frac{1}{x^\alpha}=\infty.$$

It is however continuous on any compact interval $[\varepsilon,1]$ and so it is Riemann integrable on such an interval. Note that

$$\int_\varepsilon^1 x^{-\alpha}dx = \frac{x^{1-\alpha}}{1-\alpha}\Big|_\varepsilon^1 = \frac{1}{1-\alpha}(1-\varepsilon^{1-\alpha}).$$

Since $1-\alpha>0$ we deduce that that $\varepsilon^{1-\alpha}\to 0$ as $\varepsilon\searrow 0$ and thus

$$\lim_{\varepsilon\searrow 0}\int_\varepsilon^1 x^{-\alpha}dx = \frac{1}{1-\alpha}.$$

We can define the *improper Riemann* integral of $x^{-\alpha}$ over $[0,1]$ to be

$$\int_0^1 x^{-\alpha}dx := \lim_{\varepsilon\searrow 0}\int_\varepsilon^1 x^{-\alpha}dx = \frac{1}{1-\alpha}.$$

(b) Let $p>1$ and consider the function $g:[1,\infty)\to\mathbb{R}$, $g(x)=\frac{1}{x^p}$. The function g is bounded

$$0<g(x)\leq 1,\quad \forall x\geq 1$$

but it is defined on the unbounded interval $[1,\infty)$. It is integrable on any interval $[1,L]$ and we have

$$\int_1^L x^{-p}dx = \frac{x^{1-p}}{1-p}\Big|_1^L = \frac{1}{1-p}(L^{1-p}-1).$$

Since $1-p<0$ we deduce that $L^{1-p}\to 0$ as $L\to\infty$ and thus

$$\lim_{L\to\infty}\int_1^L x^{-p}dx=-\frac{1}{1-p}=\frac{1}{p-1}.$$

We define the *improper Riemann* integral of x^{-p} over $[1,\infty)$ to be

$$\int_1^\infty x^{-p}dx:=\lim_{L\to\infty}\int_1^L x^{-p}dx=\frac{1}{p-1}.\qquad\square$$

The above examples gave meaning to integrals of *functions that are not defined on compact intervals*. Such integrals are called *improper*.

Definition 9.60 (Improper integrals)**.** (a) Let $-\infty<a<\omega\leq\infty$. Given a function $f:[a,\omega)\to\mathbb{R}$ we say that the *improper integral*

$$\int_a^\omega f(x)\,dx$$

is *convergent* if

- the restriction of f to any interval $[a,x]\subset[a,\omega)$ is Riemann integrable and,
- the limit
$$\lim_{x\nearrow\omega}\int_a^x f(t)\,dt$$
exists and it is finite.

When these happen we set

$$\int_a^\omega f(x)dx:=\lim_{x\nearrow\omega}\int_a^x f(t)\,dt.$$

(b) Let $-\infty\leq\omega<b<\infty$. Given a function $f:(\omega,b]\to\mathbb{R}$ we say that the *improper integral*

$$\int_\omega^b f(x)dx$$

is *convergent* if

- the restriction of f to any interval $[x,b]\subset(\omega,b]$ is Riemann integrable and
- the limit
$$\lim_{x\searrow\omega}\int_x^b f(t)\,dt$$
exists and it is finite.

When these happen we set

$$\int_\omega^b f(x)dx:=\lim_{x\searrow\omega}\int_x^b f(t)\,dt.\qquad\square$$

Remark 9.61. (a) We can rephrase Example 9.59(a) by saying that the integral

$$\int_0^1\frac{1}{x^\alpha}dx$$

is convergent if $\alpha \in (0,1)$. Example 9.59(b) shows that the integral

$$\int_1^\infty \frac{1}{x^p}dx$$

is convergent if $p > 1$.

(b) In the sequel, in order to keep the presentation within bearable limits, we will state and prove results only for the improper integrals of type (a) in Definition 9.60. These involve functions that have a "problem" at the *upper* endpoint ω of their domain: either that endpoint is infinite, or the function "explodes" as x approaches ω

These results have obvious counterparts for the integrals of type (b) in Definition 9.60 that involve functions that have a "problem" at the *lower* endpoint of their domain. Their statements and proofs closely mimic the corresponding ones for type (a) integrals. □

Example 9.62. For any $a, b \in \mathbb{R}$, $a < b$, the improper integrals

$$\int_a^b \frac{1}{(x-a)^\alpha}dx,\;\; \int_a^b \frac{1}{(b-x)^\alpha}dx$$

are convergent for $\alpha < 1$ and divergent if $\alpha \geq 1$. Indeed, if $\alpha \neq 1$ we have

$$\int_{a+\varepsilon}^b \frac{1}{(x-a)^\alpha}dx = \frac{1}{1-\alpha}(x-a)^{1-\alpha}\Big|_{x=a+\varepsilon}^{a=b} = \frac{1}{1-\alpha}\big((b-a)^{1-\alpha} - \varepsilon^{1-\alpha}\big),$$

If $\alpha = 1$ we have

$$\int_{a+\varepsilon}^b \frac{1}{(x-a)}dx = \ln(x-a)\Big|_{x=a+\varepsilon}^{x=b} = \ln(b-a) - \ln\varepsilon.$$

These computations show that

$$\lim_{\varepsilon\searrow 0}\int_{a+\varepsilon}^b \frac{1}{(x-a)^\alpha}dx = \begin{cases} \frac{1}{1-\alpha}(b-a)^{1-\alpha}, & \alpha < 1,\\ \infty, & \alpha \geq 1.\end{cases}$$

The convergence of the integral

$$\int_a^b \frac{1}{(b-x)^\alpha}dx$$

is analyzed in a similar fashion.

(b) The integral

$$\int_1^\infty \frac{1}{x^p}dx,\;\; p \in \mathbb{R}.$$

is convergent for $p > 1$ and divergent if $p \leq 1$.

Indeed, if $p \neq 1$, then

$$\int_1^L x^{-p}dx = \frac{1}{1-p}x^{1-p}\Big|_{x=1}^{x=L} = \frac{1}{1-p}\big(L^{1-p} - 1\big).$$

Now observe that

$$\lim_{L\to\infty} L^{1-p} = \begin{cases} 0, & p > 1,\\ \infty, & p < 1.\end{cases}$$

When $p = 1$, we have

$$\int_1^L \frac{1}{x}dx = \ln L \to \infty \;\text{ as }\; L \to \infty.$$

Similarly, the integral

$$\int_{-\infty}^{-1} \frac{1}{|x|^p} dx$$

converges for $p > 1$ and diverges for $p \leq 1$. □

We have the following immediate result whose proof is left to you as an exercise.

Proposition 9.63. *Let $-\infty < a < \omega \leq \infty$ and $f_1, f_2 : [a, \omega) \to \mathbb{R}$ be functions that are Riemann integrable on each of the intervals $[a, x]$, $x \in (a, \omega)$.*

(a) If $t_1, t_2 \in \mathbb{R}$, and the improper integrals

$$\int_a^\omega f_i(x) dx, \quad i = 1, 2$$

are convergent, then the integral

$$\int_a^\omega \big(t_1 f_1(x) + t_2 f_2(x) \big) dx$$

is convergent, and

$$\int_a^\omega \big(t_1 f_1(x) + t_2 f_2(x) \big) dx = t_1 \int_a^\omega f_1(x) dx + t_2 \int_a^\omega f_2(x) dx.$$

(b) Let $b \in (a, \omega)$. The improper integral

$$\int_a^\omega f_1(x) dx$$

is convergent if and only if the improper integral

$$\int_b^\omega f_1(x) dx$$

is convergent. Moreover, when these integrals are convergent we have

$$\int_a^\omega f_1(x) dx = \int_a^b f_1(x) dx + \int_b^\omega f_1(x) dx. \tag{9.60}$$

□

Theorem 9.64 (Cauchy)**.** *Let $-\infty < a < \omega \leq \infty$ and suppose that $f : [a, \omega) \to \mathbb{R}$ is a function which is Riemann integrable on each of the intervals $[a, x] \subset [a, \omega)$. Then the following statements are equivalent.*

(i) *The integral $\int_a^\omega f(t) dt$ is convergent.*

(ii) *For any $\varepsilon > 0$ there exists $c = c(\varepsilon) \in (a, \omega)$ such that*

$$\forall x, y : \quad x, y \in (c(\varepsilon), \omega) \Rightarrow \left| \int_x^y f(t) dt \right| < \varepsilon.$$

Proof. We set

$$I(x) := \int_a^x f(t) dt, \quad \forall x \in [a, \omega).$$

(i) $\Rightarrow$ (ii).We know that the limit

$$I_\omega := \lim_{x \to \omega} I(x)$$

exists and it is finite. Let $\varepsilon > 0$. There exists $c = c(\varepsilon) \in [a, \omega)$ such that

$$\forall x, y: \;\; x, y \in (c, \omega) \Rightarrow |I(x) - I_\omega| < \frac{\varepsilon}{2} \;\text{ and }\; |I(y) - I_\omega| < \frac{\varepsilon}{2}.$$

Observe that for any $x, y \in (c, \omega)$ we have

$$\left| \int_x^y f(t)dt \right| = \bigl|\, I(y) - I(x) \,\bigr| \leq \bigl|\, I(y) - I_\omega \,\bigr| + \bigl|\, I_\omega - I(x) \,\bigr| < \varepsilon.$$

This proves (ii).

(ii) $\Rightarrow$ (i). We know that for any $\varepsilon > 0$ there exists $c = c(\varepsilon) \in [a, \omega)$ such that

$$\forall x < y: \;\; x, y \in (\, c(\varepsilon), \omega \,) \Rightarrow \left| \int_x^y f(t)dt \right| < \frac{\varepsilon}{2}. \tag{9.61}$$

Choose a sequence (x_n) in $[a, \omega)$ such that

$$\lim_n x_n = \omega.$$

We deduce that for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that

$$\forall n: \;\; n > N(\varepsilon) \Rightarrow x_n \in (\, c(\varepsilon), \omega).$$

Hence, for any $m, n > N(\varepsilon)$ we have

$$|I(x_m) - I(x_n)| < \frac{\varepsilon}{2} < \varepsilon, \;\; \forall m, n > N(\varepsilon) \tag{9.62}$$

proving that the sequence $(I(x_n))$ is Cauchy, thus convergent. Set

$$J := \lim_n I(x_n).$$

We will show that

$$\lim_{x \to \omega} I(x) = J.$$

Letting $m \to \infty$ in (9.62) we deduce that for any $\varepsilon > 0$ and any $n > N(\varepsilon)$ we have

$$x_n \in (\, c(\varepsilon), \omega) \;\text{ and }\; |J - I(x_n)| \leq \frac{\varepsilon}{2}. \tag{9.63}$$

Let $x \in (\, c(\varepsilon), \omega)$ and $n > N(\varepsilon/2)$. Then $x, x_n \in (\, c(\varepsilon), \omega)$ and (9.61) implies that

$$|I(x_n) - I(x)| < \frac{\varepsilon}{2} \tag{9.64}$$

We deduce

$$|I(x) - J| \leq |\, I(x) - I(x_n) \,| + |\, I(x_n) - J \,| \stackrel{(9.63),(9.64)}{<} \varepsilon, \;\; \forall x \in (c(\varepsilon), \omega).$$

This proves (i). $\square$

Corollary 9.65 (Comparison Principle)**.** *Let $-\infty < a < \omega \leq \infty$ and suppose that $f, g : [a, \omega) \to \mathbb{R}$ are two real functions satisfying the following properties.*

(i) *For any $x \in [a, \omega)$ the restrictions of f, g to $[a, x]$ are Riemann integrable.*

(ii) *$\exists b \in [a, \omega)$, such that $0 \leq f(x) \leq g(x)$, $\forall x \in [b, \omega)$.*

Then

$$\int_a^\omega g(x)dx \;\textit{ is convergent } \Rightarrow \int_a^\omega f(x)dx \;\textit{ is convergent.}$$

Proof. Since the improper integral

$$\int_a^\omega g(x)dx$$

is convergent we deduce from Proposition 9.63(b) that the integral

$$\int_b^\omega g(x)dx$$

is also convergent. Theorem 9.64 shows that for any $\varepsilon > 0$ there exists $c(\varepsilon) \in [b, \omega)$ such that

$$\forall x < y: \quad x, y \in (c(\varepsilon), \omega) \Rightarrow \int_x^y g(t)dt = \left| \int_x^y g(t)dt \right| < \varepsilon.$$

Using the assumption (i) we deduce that

$$\forall x < y: \quad x, y \in (c(\varepsilon), \omega) \Rightarrow \left| \int_x^y f(t)dt \right| = \int_x^y f(t)dt \le \int_x^y g(t)dt.$$

We can invoke Theorem 9.64 to conclude that the integral

$$\int_b^\omega f(x)dx$$

is convergent. Proposition 9.63(b) now implies that

$$\int_a^\omega f(x)dx$$

is convergent. □

Remark 9.66. Using the logical tautology

$$p \Rightarrow q \longleftrightarrow \neg q \Rightarrow \neg p,$$

we see that if f and g are as in Corollary 9.65, then

$$\int_a^\omega f(x)dx \text{ is divergent } \Rightarrow \int_a^\omega g(x)dx \text{ is divergent.}$$ □

Corollary 9.67. *Let $-\infty < a < \omega \le \infty$ and suppose that $f, g : [a, \omega) \to \mathbb{R}$ are two real functions satisfying the following properties.*

(i) $\exists b \in [a, \omega)$, *such that* $f(x) \ge 0$ *and* $g(x) > 0$, $\forall x \in [b, \omega)$.

(ii) *There exists* $C \ge 0$ *such that*

$$\lim_{x \to \omega} \frac{f(x)}{g(x)} = C.$$

(iii) *For any* $x \in [a, \omega)$ *the restrictions of* f *and* g *to* $[a, x]$ *are Riemann integrable.*

Then

$$\int_a^\omega g(x)dx \text{ is convergent } \Rightarrow \int_a^\omega f(x)dx \text{ is convergent.}$$

Proof. The integral

$$\int_a^\omega (C+1)g(x)dx$$

is convergent.

The assumption (ii) implies that there exists $b_0 \in (b, \omega)$ such that

$$f(x) < (C+1)g(x), \ \ \forall x \in (b_0, \omega).$$

We can now invoke Corollary 9.65 to reach the desired conclusion. □

Example 9.68. (a) Consider the continuous function

$$f : [1, \infty) \to \mathbb{R}, \ \ f(x) = \frac{x+2}{4x^3+3x^2+2x+1}$$

Note that $f(x) \geq 0$ for any $x \in [1, \infty)$. To decide the convergence of the integral

$$\int_1^\infty f(x)dx$$

we compare $f(x)$ with the function $g : [1, \infty) \to \mathbb{R}$, $g(x) = \frac{1}{x^2}$. Observe that

$$\frac{f(x)}{g(x)} = \frac{x^3+2x^2}{4x^3+3x^2+2x+1} \to \frac{1}{4} \ \text{ as } x \to \infty$$

Since

$$\int_1^\infty \frac{1}{x^2}dx$$

is convergent we deduce from Corollary 9.67 that the integral

$$\int_1^\infty f(x)dx$$

is also convergent.

(b) Consider the function

$$f : (0, 1] \to \mathbb{R}, \ \ f(x) = \frac{\sin\sqrt{x}}{x}.$$

Note that

$$\lim_{x \searrow 0} f(x) = \lim_{x \searrow 0} \frac{\sin\sqrt{x}}{\sqrt{x}} \frac{1}{\sqrt{x}} = \infty.$$

In particular, $f(x) > 0$ for $x > 0$ small. Since

$$\frac{f(x)}{\frac{1}{\sqrt{x}}} = \frac{\sin\sqrt{x}}{\sqrt{x}} \to 1 \ \text{ as } x \searrow 0$$

and the improper integral

$$\int_0^1 \frac{1}{\sqrt{x}}dx$$

is convergent, we deduce from Corollary 9.67 that the improper integral $\int_0^1 f(x)dx$ is also convergent.

(c) Consider the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = xe^{-x^2}$. Note that $f(x) \geq 0$, $\forall x$ and

$$\frac{f(x)}{\frac{1}{x^2}} = x^3e^{-x^2} = \frac{x^2}{e^{x^2}} \to 0 \ \text{ as } x \to \infty.$$

Thus the integral

$$\int_0^\infty xe^{-x^2}dx$$

is convergent. To evaluate this integral we begin by evaluating the integrals

$$\int_0^L xe^{-x^2}dx$$

where $L \to \infty$. We use the change in variables $u = x^2$ so that $du = 2xdx$

$$x = 0 \Rightarrow u = 0, \;\; x = L \Rightarrow u = L^2$$

and we deduce

$$\int_0^L xe^{-x^2}dx = \frac{1}{2}\int_0^L e^{-x^2}(2xdx) = \frac{1}{2}\int_0^{L^2} e^{-u}du = \frac{1}{2}\Big(-e^{-u}\Big)\Big|_{u=0}^{u=L^2} = \frac{1}{2}(1-e^{-L^2}).$$

Now observe that

$$\lim_{L\to\infty}\frac{1}{2}(1-e^{-L^2}) = \frac{1}{2},$$

so that

$$\int_0^\infty xe^{-x^2}dx = \frac{1}{2}.$$

So far we have investigated improper integrals of function that had a problem at ω, one of the endpoints of its domain: either $\omega = \infty$, or the function "explodes" as it approaches ω. Sometime we need to deal with functions that have problems at both endpoints of its domain. The next example explains how to proceed in this case.

(d) Consider the function

$$f : (-1, 1) \to \mathbb{R}, \;\; f(x) = \frac{1}{\sqrt{(1-x^2)}}.$$

To decide the convergence of the integral

$$\int_{-1}^1 f(x)dx,$$

we must first locate the sources of the possible problems. We note that $f(x)$ "explodes" as $x \to \pm 1$, i.e.,

$$\lim_{x\to\pm1} f(x) = \infty.$$

We split the integral into two parts,

$$I_{-1} = \int_{-1}^0 f(x)dx, \;\; I_1 = \int_0^1 f(x)dx.$$

Each of the above integrals has only one problem point and, if both integrals are convergent, then the original integral will be convergent if and only if both integrals above are convergent and, when this happens, we have

$$\int_{-1}^1 f(x)dx = \int_{-1}^0 f(x)dx + \int_0^1 f(x)dx.$$

Now observe that

$$f(x) = \frac{1}{\sqrt{(1-x)(1+x)}}.$$

The term $(1-x)$ is responsible for the bad behavior near $x=1$, while the term $(1+x)$ is responsible for the bad behavior near $x=-1$.

From Example 9.62 we deduce that both integrals

$$\int_{-1}^{0}\frac{1}{\sqrt{1+x}}dx,\ \ \int_0^1\frac{1}{\sqrt{1-x}}$$

are convergent. Observe next that

$$\lim_{x\to -1}\frac{f(x)}{\frac{1}{\sqrt{1+x}}}=\lim_{x\to -1}\frac{\frac{1}{\sqrt{(1-x)(1+x)}}}{\frac{1}{\sqrt{1+x}}}=\lim_{x\to -1}\frac{1}{\sqrt{1-x}}=\frac{1}{\sqrt{2}},$$

$$\lim_{x\to 1}\frac{f(x)}{\frac{1}{\sqrt{1-x}}}=\lim_{x\to 1}\frac{\frac{1}{\sqrt{(1-x)(1+x)}}}{\frac{1}{\sqrt{1-x}}}=\lim_{x\to 1}\frac{1}{\sqrt{1+x}}=\frac{1}{\sqrt{2}}.$$

Using Corollary 9.67 we now deduce that both integrals $I_{\pm 1}$ are convergent. In particular, we deduce that the improper integral

$$\int_{-1}^{1}f(x)dx$$

is convergent. We can actually compute it. Let $-1<a<0<b<1$. We have

$$\int_a^b\frac{1}{\sqrt{1-x^2}}dx=\arcsin x\Big|_{x=a}^{x=b}=\arcsin b-\arcsin a.$$

Note that

$$\lim_{b\nearrow 1}\arcsin b=\arcsin 1=\frac{\pi}{2},\ \ \lim_{a\searrow -1}\arcsin a=\arcsin(-1)=-\frac{\pi}{2}$$

so that

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}dx=\frac{\pi}{2}-\Big(-\frac{\pi}{2}\Big)=\pi. \tag{9.65}$$

Definition 9.69. Let $-\infty<a<\omega\leq\infty$ and $f:[a,\omega)\to\mathbb{R}$ a function that is Riemann integrable on any interval $[a,x]$, $x\in(a,\omega)$. We say that the improper integral

$$\int_a^\omega f(x)dx$$

is *absolutely convergent* if the improper integral

$$\int_a^\omega |f(x)|dx$$

is convergent. □

The next result is very similar to Theorem 4.46.

Proposition 9.70. *Let $-\infty<a<\omega\leq\infty$ and $f:[a,\omega)\to\mathbb{R}$ a function that is Riemann integrable on any interval $[a,x]$, $x\in(a,\omega)$. Then*

$$\int_a^\omega f(x)dx\ \ \textit{absolutely convergent}\ \Rightarrow\ \int_a^\omega f(x)dx\ \ \textit{convergent.}$$

Proof. We rely on Cauchy's Theorem 9.64. Since the integral

$$\int_a^\omega |f(x)|dx$$

is convergent we deduce from Cauchy's theorem that for any $\varepsilon > 0$ there exists $c(\varepsilon) \in (a, \omega)$ such that

$$\forall x, y;\ \ x, y \in (c(\varepsilon), \omega) \Rightarrow \left|\int_x^y |f(t))|dt\right| < \varepsilon.$$

On the other hand, (9.28) shows that

$$\left|\int_x^y f(t)dt\right| \leq \left|\int_x^y |f(t)|dt\right|$$

and we deduce that

$$\forall x, y,\ \ x, y \in (c(\varepsilon), \omega) \Rightarrow \left|\int_x^y f(t))dt\right| < \varepsilon.$$

Cauchy's theorem now implies that

$$\int_a^\omega f(x)dx$$

is convergent. □

The comparison principle Corollary 9.65 yields a comparison principle involving absolute convergence.

Corollary 9.71 (Comparison Principle)**.** *Let $-\infty < a < \omega \leq \infty$ and suppose that $f, g : [a, \omega) \to \mathbb{R}$ are two real functions satisfying the following properties.*

(i) *$\exists b \in [a, \omega)$, such that $|f(x)| \leq |g(x)|$, $\forall x \in [b, \omega)$.*

(ii) *For any $x \in [a, \omega)$ the restrictions of f, g to $[a, x]$ are Riemann integrable.*

Then

$$\int_a^\omega g(x)dx \ \text{ is absolutely convergent } \Rightarrow \int_a^\omega f(x)dx \ \text{ is absolutely convergent.}$$

□

Example 9.72. Consider the function

$$f : [1, \infty) \to \mathbb{R},\ \ f(x) = \frac{\sin x}{x^2}.$$

Note that

$$|f(x)| \leq \frac{1}{x^2},\ \ \forall x \geq 1$$

and since $\int_1^\infty \frac{1}{x^2}dx$ is convergent we deduce that $\int_a^\infty f(x)dx$ is absolutely convergent. □

9.7.1. The Euler's Gamma and Beta functions. For every $x > 0$ we set

$$\boxed{\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt}. \tag{9.66}$$

For each fixed $x > 0$ this improper integral is convergent. To se this we split the above integral into two parts

$$I_0 = \int_0^1 t^{x-1}e^{-t}dt, \;\; I_\infty = \int_1^\infty t^{x-1}e^{-t}dt.$$

To prove the convergence of I_0 we observe that

$$0 < t^{x-1}e^{-t} \leq t^{x-1} \;\; \forall t \in (0,1].$$

Since $x - 1 > -1$ the improper integral

$$\int_0^1 t^{x-1}dt$$

is convergent. The Comparison Principle then implies that I_0 is also convergent.

To prove the convergence of I_∞ we observe that and as $t \to \infty$ the function $t^{x-1}e^{-t}$ decays to zero faster, than any power t^{-n}, $n \in \mathbb{N}$. In particular

$$\lim_{t\to\infty} \frac{t^{x-1}e^{-t}}{t^{-2}}dt = 0.$$

Since the integral

$$\int_1^\infty t^{-2}dt$$

is convergent we deduce from the Comparison Principle that I_∞ is convergent as well.

The resulting function

$$(0,\infty) \ni x \mapsto \Gamma(x) \in (0,\infty)$$

is called *Euler's Gamma function*

Observe that

$$\Gamma(1) = \int_0^\infty e^{-t}dt = \big(-e^{-t}\big)\Big|_{t=0}^{t=\infty} = 1, \tag{9.67}$$

and, for $x > 0$,

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}dt = -\int_0^\infty t^x d(e^{-t}) = -\underbrace{\big(t^x e^{-t}\big)\Big|_{t=0}^{\infty}}_{=0} + x\underbrace{\int_0^\infty t^{x-1}e^{-t}dt}_{=\Gamma(x)} = x\Gamma(x).$$

so that

$$\boxed{\Gamma(x+1) = x\Gamma(x), \;\; \forall x > 0.} \tag{9.68}$$

From (9.67) and (9.68) we deduce inductively

$$\Gamma(2) = 1\Gamma(1) = 1, \;\; \Gamma(3) = 2\Gamma(2) = 2, \;\; \Gamma(4) = 3\Gamma(3) = 3\cdot 2 = 3!, \dots,$$

$$\boxed{\Gamma(n) = (n-1)!, \;\; \forall n \in \mathbb{N}.} \tag{9.69}$$

Fix $\lambda > 0$. In the definition

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

we make the change of variables $t = \lambda s$ we deduce

$$\Gamma(x) = \int_0^\infty \lambda^{x-1} s^{x-1} e^{-\lambda s} \lambda ds = \lambda^x \int_0^\infty s^{x-1} e^{-\lambda s} ds,$$

so that

$$\boxed{\frac{\Gamma(x)}{\lambda^x} = \int_0^\infty s^{x-1} e^{-\lambda s} ds, \quad \forall x, \lambda > 0.} \tag{9.70}$$

For $x, y > 0$ we set

$$\boxed{B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt.} \tag{9.71}$$

This integral is convergent since $x-1, y-1 > -1$. The resulting function

$$(0,\infty) \times (0,\infty) \ni (x,y) \mapsto B(x,y) \in (0,\infty),$$

is known as *Euler's Beta function*.

Some values of this function can be computed explicitly. For example

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 \frac{1}{\sqrt{t(1-t)}} dt.$$

If we make the change in variables $t = s^2$, $s \geq 0$, then $s = 0$ when $t = 0$, $s = 1$ when $t = 1$, $dt = 2sds$ and we deduce

$$\int_0^1 \frac{1}{\sqrt{t(1-t)}} dt = \int_0^1 \frac{1}{\sqrt{s(1-s^2)}} 2sds = 2\int_0^1 \frac{1}{\sqrt{1-s^2}} ds = 2\Big(\arcsin s\Big|_{s=0}^{s=1}\Big) = \pi.$$

Hence

$$\boxed{B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.} \tag{9.72}$$

If we make the change in variables in the integral (9.71)

$$u = \frac{t}{1-t},$$

then we observe that $u = 0$ when $t = 0$ and $u \to \infty$ as $t \nearrow 1$. Moreover, we have

$$(1-t)u = t \Rightarrow u = t(1+u) \Rightarrow t = \frac{u}{1+u} = 1 - \frac{1}{1+u} \Rightarrow 1-t = \frac{1}{1+u}, \quad dt = \frac{1}{(1+u)^2} du,$$

$$t^{x-1}(1-t)^{y-1} dt = \left(\frac{u}{1+u}\right)^{x-1} \left(\frac{1}{1+u}\right)^{y-1} \frac{1}{(1+u)^2} du = \frac{u^{x-1}}{(1+u)^{x+y}} du,$$

so that

$$\boxed{B(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} du, \quad \forall x, y > 0.} \tag{9.73}$$

Using (9.70) we deduce

$$\frac{1}{(1+u)^{x+y}} = \frac{1}{\Gamma(x+y)} \int_0^\infty s^{x+y-1} e^{-(1+u)s} ds.$$

Using this in (9.73) we deduce

$$\begin{aligned} B(x,y) &= \frac{1}{\Gamma(x+y)}\int_0^\infty u^{x-1}\left(\int_0^\infty s^{x+y-1}e^{-(1+u)s}ds\right)du \\ &= \frac{1}{\Gamma(x+y)}\underbrace{\int_0^\infty\left(\int_0^\infty u^{x-1}s^{x+y-1}e^{-(1+u)s}ds\right)du}_{=:\boldsymbol{I}}. \end{aligned} \tag{9.74}$$

The above complicated looking integrals are called *iterated integrals*. At this point we want to invoke without proof[1] *Fubini's theorem* which allows us to conclude that we can interchange the order of integration so that

$$\boldsymbol{I} := \int_0^\infty\left(\int_0^\infty u^{x-1}s^{x+y-1}e^{-(1+u)s}ds\right)du = \int_0^\infty\left(\int_0^\infty s^{x+y-1}u^{x-1}e^{-(1+u)s}du\right)ds$$

$$= \int_0^\infty s^{x+y-1}\left(\int_0^\infty u^{x-1}e^{-(1+u)s}du\right)ds = \int_0^\infty s^{x+y-1}\left(\int_0^\infty u^{x-1}e^{-s}e^{-su}du\right)ds$$

$$= \int_0^\infty e^{-s}s^{x+y-1}\underbrace{\left(\int_0^\infty u^{x-1}e^{-su}du\right)}_{\overset{(9.70)}{=}\frac{\Gamma(x)}{s^x}}ds$$

$$= \int_0^\infty e^{-s}s^{x+y-1}\frac{\Gamma(x)}{s^x}ds = \Gamma(x)\int_0^\infty s^{y-1}e^{-s}ds = \Gamma(x)\Gamma(y).$$

Thus

$$\boldsymbol{I} = \Gamma(x)\Gamma(y),$$

Using the last equality in (9.74) we conclude that

$$\boxed{B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ \ \forall x,y>0.} \tag{9.75}$$

Using this in (9.72) we deduce

$$\pi = B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \Gamma\left(\frac{1}{2}\right)^2.$$

Hence

$$\sqrt{\pi} = \Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}dt.$$

If in the last integral we make the change in variables $t = s^2$, then we deduce

$$\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt = 2\int_0^\infty e^{-s^2}ds.$$

so that

$$\boxed{\int_0^\infty e^{-s^2}ds = \frac{\sqrt{\pi}}{2}.} \tag{9.76}$$

[1]The proof is nontrivial and requires techniques of real analysis in higher dimensional spaces.

9.8. Length, area and volume

The concept of integral is involved in the definition of important geometric quantities such length, area and volume. Their definition in the most general context is quite involved and we restrict ourselves to special cases that still have a wide range of applications.

9.8.1. Length. We will define the length of special curves in the plane, namely the curves defined by the graphs of differentiable functions.

Definition 9.73. Suppose that $-\infty \leq a < b \leq \infty$ and $f : (a,b) \to \mathbb{R}$ is a C^1-function. We say that its graph has *finite length* if the integral

$$\int_a^b \sqrt{1+f'(x)^2}dx$$

is convergent. The value of this integral is then declared to be the *length of the graph of* f.□

Here is the intuition behind the definition. If we are located at the point $(x_0, y_0) = (x_0, f(x_0))$ on the graph of f and we move a tiny bit, from x_0 to $x_0 + dx$, then the rise, that is is the change in altitude is

$$dy = \frac{dy}{dx} \cdot dx = f'(x_0)dx.$$

The Pythagorean theorem then shows that the distance covered along the graph is approximately

$$\sqrt{dx^2 + dy^2} = \sqrt{dx^2 + f'(x_0)^2 dx^2} = \sqrt{1+f'(x_0)^2}\, dx.$$

The total distance traveled along the graph, i.e., the length of the trap is obtained by summing all these infinitesimal distances

$$\int_a^b \sqrt{1+f'(x)^2}\, dx.$$

The next examples support the validity of the proposed formula for the length.

Example 9.74. Consider two points in the plane, P_1 with coordinates (x_1, y_1) and P_2 with coordinates (x_2, y_2). Assume moreover that $x_1 < x_2$; see Figure 9.6. We want to compute the length $|P_1P_2|$ of the line segment connecting P_1 to P_2.

Pythagoras' theorem shows that

$$|P_1P_2|^2 = |P_1Q|^2 + |QP_2|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2. \tag{9.77}$$

Let us show that the formula proposed in Definition 9.73 yields the same result.

The line determined by the points P_1, P_2 has slope

$$m := \frac{y_2 - y_1}{x_2 - x_1},$$

and thus it is described by the equation

$$y = m(x - x_1) + y_1.$$

In other words, the line segment is the graph of the linear function

$$f : [x_1, x_2] \to \mathbb{R},\ \ f(x) = m(x - x_1) + y_1.$$

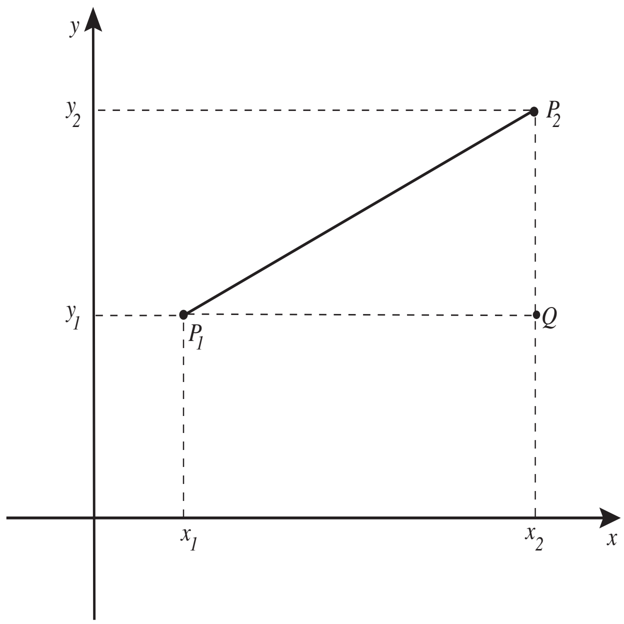

Figure 9.6. *Computing the length of a line segment.*

Note that $f'(x) = m$, $\forall x \in [x_1, x_2]$ and, according to Definition 9.73, we have

$$|P_1P_2| = \int_{x_1}^{x_2} \sqrt{1 + f'(x)^2}dx = \int_{x_1}^{x_2} \sqrt{1 + m^2}dx = \sqrt{1 + m^2}(x_2 - x_1).$$

Hence

$$|P_1P_2|^2 = (1 + m^2)(x_2 - x_1)^2 = \left(1 + \frac{(y_2 - y_1)^2}{(x_2 - x_1)^2}\right)(x_2 - x_1)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

This agrees with the Pythagorean prediction (9.77). □

Example 9.75. Consider the function

$$f : (-1, 1) \to \mathbb{R}, \;\; f(x) = \sqrt{1 - x^2}.$$

The graph of this function is the upper half-circle of radius 1 centered at the origin; see Figure 9.7. Indeed, a point (x, y) on this circle satisfies

$$x^2 + y^2 = 1, \;\; y \geq 0 \Longleftrightarrow y = \sqrt{1 - x^2}.$$

The function $f(x)$ is differentiable on $(-1, 1)$ and we have

$$f'(x) = -\frac{x}{\sqrt{1 - x^2}}, \;\; 1 + f'(x)^2 = 1 + \frac{x^2}{1 - x^2} = \frac{1}{1 - x^2}, \;\; \forall x \in (-1, 1). \tag{9.78}$$

Hence the length of this semi-circle is

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}dx \stackrel{(9.65)}{=} \pi.$$

□

We can define the length of more complicated curves.

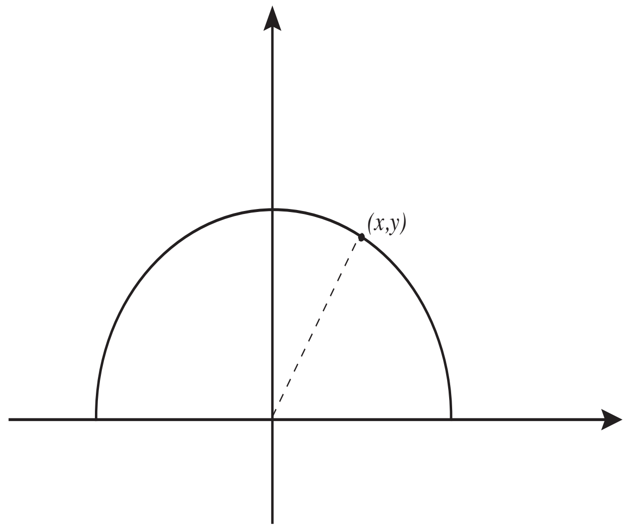

Figure 9.7. *Computing the length of a half-circle.*

Definition 9.76. Let $-\infty < a < b \leq \infty$. A *continuous* function $(a, b) \to \mathbb{R}$ is called *piecewise* C^1 if there exist points $x_1, \dots, x_n \in (a, b)$ such that

$$a < x_1 < x_2 < \cdots < x_n < b$$

and the function f is C^1 on each of the subintervals

$$(a, x_1), \ \ (x_1, x_2), \dots, (x_n, b).$$

The length of its graph is then given by

$$\int_a^{x_1} \sqrt{1 + f'(x)^2} dx + \int_{x_1}^{x_2} \sqrt{1 + f'(x)^2} dx + \cdots + \int_{x_n}^{b} \sqrt{1 + f'(x)^2} dx.$$

Above, some of the integrals could be improper and for the length to be finite these integrals have to be convergent. □

9.8.2. Area. A region D of the cartesian plane $\mathbb{R}^2$ is said to be of *simple type* with respect to the x-axis if there exists an interval I and functions

$$F, C : I \to \mathbb{R}$$

such that

$$F(x) \leq C(x), \ \ \forall x \in I,$$

and

$$(x, y) \in D \iff x \in I \ \wedge \ F(x) \leq y \leq C(x).$$

The function F is called the *floor* of the region D, while the function C is called the *ceiling* of the region; see Figure 9.8

The *area* of the region D is given by the improper integral

$$\operatorname{Area}(D) := \int_I \big(C(x) - F(x) \big) \, dx,$$

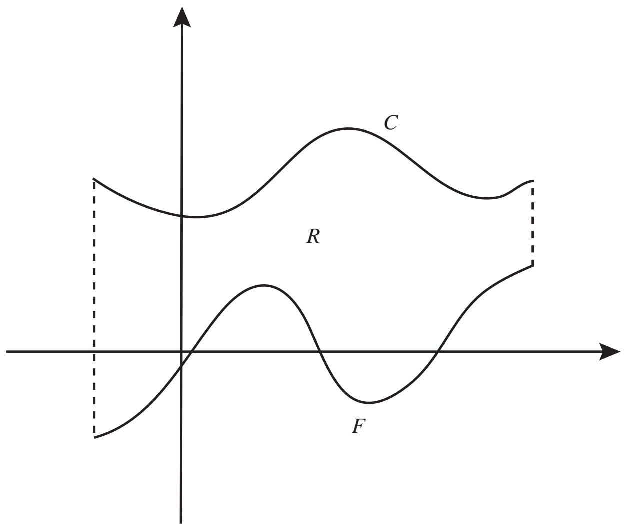

Figure 9.8. *A planar region of simple type with respect to the x-axis.*

whenever this integral is well defined[2] and convergent.

A region D of the cartesian plane $\mathbb{R}^2$ is said to be of *simple type* with respect to the y-axis if there exists an interval J and functions

$$L, R : J \to \mathbb{R}$$

such that

$$L(y) \leq R(y), \ \ \forall y \in J$$

and

$$(x,y) \in R \Longleftrightarrow y \in J \ \wedge \ L(y) \leq x \leq R(y).$$

The function F is called the *left wall* of the region D, while the function R is called the *right wall* of the region; see Figure 9.9.

The *area* of the region D is given by the improper integral

$$\mathrm{Area}\,(D) := \int_J \big(\, R(y) - L(y)\,\big)\, dy,$$

whenever this integral is well defined

Remark 9.77. (a) We swept under the rug a rather subtle fact. A region in the plane can be simultaneously simple type with respect to the x-axis, and simple type with respect to the y-axis. In such situations there are two possible ways of computing the area and they'd better produce the same result. This is indeed the case, but the proof in general is quite complicated, and the best approach relies on the concept of multiple integrals.

To see that this is not merely a theoretical possibility, consider the region (see Figure 9.10)

$$R = \big\{ (x,y) \in \mathbb{R}^2; \ \ x \in [0,1], \ \ x^2 \leq y \leq x \,\big\}.$$

[2]The integral is well defined if the function $C(x) - F(x)$ is Riemann integrable on any compact interval $[\alpha, \beta] \subset (a,b)$.

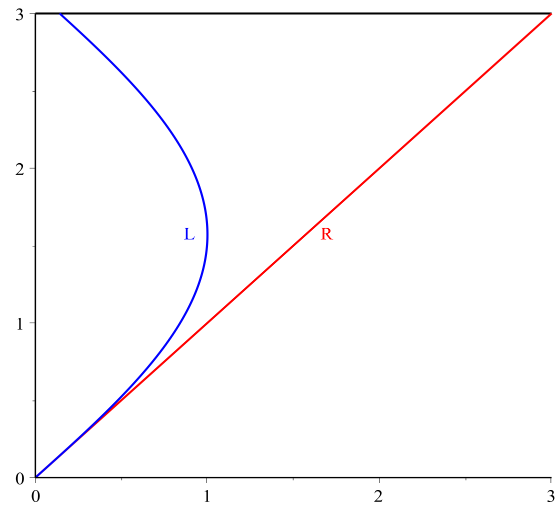

Figure 9.9. *A planar region of simple type with respect to the y-axis,* $\sin y \leq x \leq y$, $0 \leq y \leq 3$.

The above description shows that R is a region of simple type with respect to the x-axis. However, R can be given the alternate description as a region of simple type with respect to the y-axis,

$$R = \left\{(x,t) \in \mathbb{R}^2;\ \ y \in [0,1],\ \ y \leq x \leq \sqrt{y}\,\right\}.$$

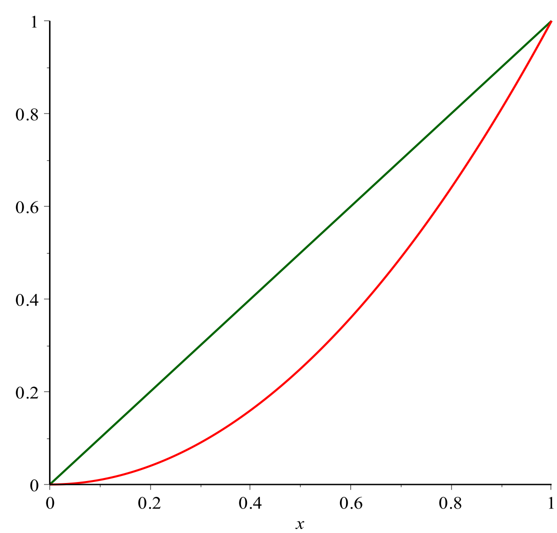

Figure 9.10. *A planar region that simple type with respect to both axes:* $x^2 \leq y \leq x$, $0 \leq x \leq 1$.

If we use the first description we deduce

$$\text{Area}\,(R) = \int_0^1 (x - x^2)dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

If we use the second description we deduce

$$\operatorname{Area}(R) = \int_0^1 (\sqrt{y} - y)dy = \left(\frac{2x^{3/2}}{3} - \frac{x^2}{2}\right)\Big|_{x=0}^{x=1} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

Many regions in the plane decompose into finitely many simple type regions that have overlaps only along boundary curves. For such a region, the area is defined as the sum of the areas of the simple-type sub-regions it decomposes into. This raises an even trickier question: why is the answer independent of the procedure we use to decompose the region into simple-type sub-regions? To answer this question one needs the full apparatus of multiple integrals.

(b) Let us observe that a simple-type region can have finite area, even if it is unbounded. Consider for example the region between the x-axis and the graph of the function

$$g : [0, \infty) \to \mathbb{R}, \;\; g(x) = e^{-x}.$$

The area of this region is

$$\int_0^\infty e^{-x}dx = \left(-e^{-x}\right)\Big|_0^\infty = -e^{-\infty} - (-1) = 0 + 1 = 1. \qquad \square$$

9.8.3. Solids of revolution. Suppose that we are given an open interval (a, b) and a function

$$g : (a, b) \to \mathbb{R}$$

called *generatrix* such that $g(x) \geq 0$, $\forall x \in (a, b)$. If we rotate the graph of g about the x-axis we get a surface of revolution Σ_g that surrounds a solid of revolution S_g; see Figure 9.11.

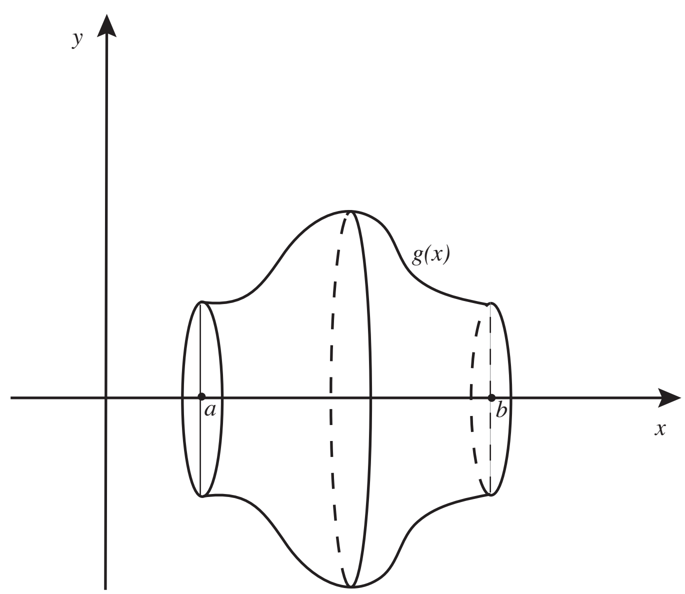

Figure 9.11. *A surface of revolution.*

The *area* of the surface of revolution Σ_g is given by the improper integral

$$\operatorname{Area}(\Sigma_g) := 2\pi \int_a^b g(x)\sqrt{1 + g'(x)^2}dx,$$

whenever the integral is well defined. The *volume* of the solid of revolution S_g is given by the improper integral

$$\mathrm{Vol}\,(S_g) := \pi \int_a^b g(x)^2 dx,$$

whenever the integral is well defined.

Example 9.78. (a) Suppose that the generatrix is the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = \sqrt{1 - x^2}$. Its graph is the upper half-circle of radius 1 depicted in Figure 9.7. When we rotate this half-circle about the x-axis, the surface of revolution obtained is a sphere Σ_g of radius 1 that surrounds a solid ball S_g of radius 1.

The computations in (9.78) show that

$$\sqrt{1 + g'(x)^2} = \frac{1}{\sqrt{1 - x^2}}$$

so that

$$g(x)\sqrt{1 + g'(x)^2} = 1.$$

We deduce that the area of the unit sphere is

$$2\pi \int_{-1}^{1} g(x)\sqrt{1 + g'(x)^2} dx = 2\pi \in_{-1}^{1} dx = 4\pi.$$

The volume of the unit ball is

$$\pi \int_{-1}^{1} g(x)^2 dx = \pi \int_{-1}^{1} (1 - x^2) dx = \pi \left(x\Big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1} \right) = \pi\Big(2 - \frac{2}{3}\Big) = \frac{4\pi}{3}.$$

These equalities confirm the classical formulæ taught in elementary solid geometry.

(b)

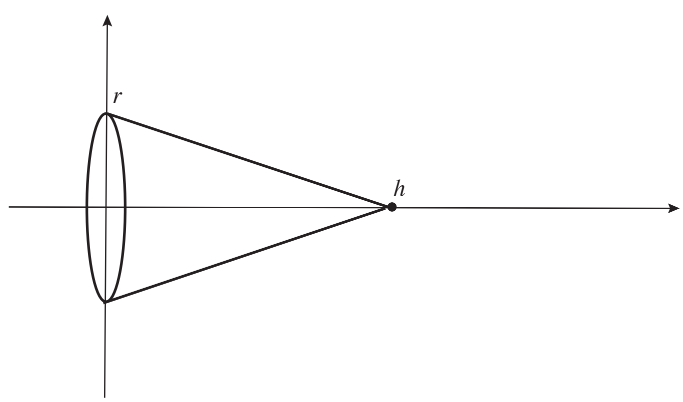

Figure 9.12. *A cone.*

Consider the cone depicted in Figure 9.12. It is obtained by rotating a line segment about the x-axis, more precisely, the line segment connecting the point $(0, r)$ on the y-axis with the point $(h, 0)$ on the x-axis. Here $h, r > 0$.

This line segment lies on the line with slope $m = -r/h$ and y-intercept r. In other words, this line is given by the equation

$$g(x) = -\frac{r}{h}x + r.$$

Observe that

$$g'(x) = -\frac{r}{h}, \quad \sqrt{1+g'(x)^2} = \frac{\sqrt{h^2+r^2}}{h},$$

$$g(x)\sqrt{1+g'(x)^2} = \frac{h^2+r^2}{h}\Big(-\frac{r}{h}x+r\Big).$$

We deduce that the area of this cone (excluding its base) is

$$2\pi\int_0^h \frac{\sqrt{h^2+r^2}}{h}\Big(-\frac{r}{h}x+r\Big)dx = 2\pi\frac{\sqrt{h^2+r^2}}{h}\int_0^h\Big(-\frac{r}{h}x+r\Big)dx$$

$$= 2\pi r\frac{\sqrt{h^2+r^2}}{h}\int_0^h dx - 2\pi r\frac{\sqrt{h^2+r^2}}{h^2}\int_0^h xdx$$

$$= 2\pi r\sqrt{h^2+r^2} - \pi r\sqrt{h^2+r^2} = \pi r\sqrt{h^2+r^2}.$$

This agrees with the known formulæ in solid geometry.

The volume of the cone is

$$\pi\int_0^h\Big(-\frac{r}{h}x+r\Big)^2dx = \frac{\pi r^2}{h^2}\int_0^h(h-x)^2dx = \frac{\pi r^2}{h^2}\times\frac{h^3}{3} = \frac{\pi r^2 h}{3}.$$

(c) Let $\alpha\in(\frac{1}{2},1)$ and consider the function

$$g:[1,\infty)\to\mathbb{R},\quad g(x)=\frac{1}{x^\alpha}.$$

The surface of revolution obtained by rotating the graph of g about the x-axis has the bugle shape in Figure 9.13

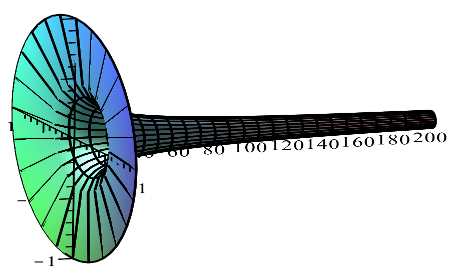

Figure 9.13. *An infinite bugle.*

The volume of this bugle is

$$\pi\int_1^\infty g(x)^2dx = \pi\int_1^\infty\frac{1}{x^{2\alpha}}dx.$$

Since $2\alpha>1$, the above integral is convergent and in fact

$$\pi\int_1^\infty\frac{1}{x^{2\alpha}}dx = \frac{\pi}{2\alpha-1}.$$

On the other hand, the area of the bugle is

$$2\pi \lim_{L\to\infty} \int_1^L g(x)\sqrt{1+g'(x)^2}dx \geq 2\pi \lim_{L\to\infty} \int_1^L g(x)dx$$
$$= 2\pi \lim_{L\to\infty} \int_1^L \frac{1}{x^\alpha}dx = 2\pi \lim_{L\to\infty} \left(\frac{x^{1-\alpha}}{1-\alpha}\right)\Big|_1^L = \infty,$$

because $\alpha < 1$. This is surprising: you need a finite amount of water to fill the bugle, but an infinite amount of paint if you want to paint it!!! □

9.9. Exercises

Exercise 9.1. Prove by induction the equality (9.3). □

Exercise 9.2. Consider the function $f : [0,4] \to \mathbb{R}$, $f(x) = x^2$, and the partition

$$\boldsymbol{P} = (0,\ 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5,\ 4)$$

of the interval $[0,4]$.

(a) Find the mesh size $\|\boldsymbol{P}\|$ of $\boldsymbol{P}$.

(b) Compute the Riemann sum $\boldsymbol{S}(f, \boldsymbol{P}, \underline{\xi})$ when the sample $\underline{\xi}$ consists of the right endpoints of the subintervals of $\boldsymbol{P}$. □

Exercise 9.3. Consider the function $f : [-2,2] \to \mathbb{R}$, $f(x) = x^2$ and the partition

$$\boldsymbol{P} = (-2,\ -1.5,\ -1\ , -0.5,\ 0,\ 0.5,\ 1,\ 1.5,\ 2)$$

of $[-2,2]$.

(a) Compute the upper and lower Darboux sums $\boldsymbol{S}^*(f,\boldsymbol{P})$, $\boldsymbol{S}_*(f,\boldsymbol{P})$.

(b) Compute $\omega(f,\boldsymbol{P})$. □

Exercise 9.4. (a) Suppose that $f, g : [a,b] \to \mathbb{R}$ are two functions. Prove that for any sampled partition $(\boldsymbol{P}, \underline{\xi})$ of $[a,b]$ and for any real numbers α, β we have

$$\boldsymbol{S}\big(\alpha f + \beta g, \boldsymbol{P}, \underline{\xi}\big) = \alpha \boldsymbol{S}\big(f, \boldsymbol{P}, \underline{\xi}\big) + \beta \boldsymbol{S}\big(g, \boldsymbol{P}, \underline{\xi}\big). \qquad \square$$

(b) Let $f : [a,b] \to \mathbb{R}$. Prove that the following statements are equivalent.

(i) The function f is *not* Riemann integrable.

(ii) There exists ε_0 such that, for any $n \in \mathbb{N}$ there exists sampled partitions $(\boldsymbol{P}_n, \underline{\xi}^n)$ and $(\boldsymbol{Q}_n, \underline{\zeta}^n)$ satisfying

$$\|\boldsymbol{P}_n\|,\ \|\boldsymbol{Q}_n\| < \frac{1}{n} \ \text{ and } \ \big|\, \boldsymbol{S}(f, \boldsymbol{P}_n, \underline{\xi}^n) - \boldsymbol{S}(f, \boldsymbol{Q}_n, \underline{\zeta}^n) \,\big| > \varepsilon_0.$$

□

Exercise 9.5. Suppose that $f : [0,1] \to \mathbb{R}$ is a C^1-function, i.e., it is differentiable on $[0,1]$ and the derivative is continuous. We set

$$M := \sup_{x \in [0,1]} |f'(x)|.$$

(a) Suppose that $I \subset [0,1]$ is an interval of length δ. Show that

$$\operatorname{osc}(f, I) \leq M\delta.$$

(b) For $n \in \mathbb{N}$ we denote by $\boldsymbol{U}_n$ the uniform partition of order n of $[0,1]$. Show that

$$\omega(f, \boldsymbol{U}_n) \leq \frac{M}{n},\ \ \forall n \in \mathbb{N}.$$

(c) Fix $n \in \mathbb{N}$ and a sample $\underline{\xi}$ of $\boldsymbol{U}_n$. Show that

$$\left| \int_0^1 f(x)dx - \boldsymbol{S}(f, \boldsymbol{U}_n, \underline{\xi}) \right| \leq \frac{M}{n}. \qquad \square$$

Exercise 9.6. Let $a > 0$ and assume that $f : [-a, a] \to \mathbb{R}$ is a Riemann integrable function.

(a) Prove that if f is an *odd* function, i.e., $f(-x) = -f(x)$, $\forall x \in [-a, a]$, then

$$\int_{-a}^{a} f(x)dx = 0.$$

(b) Prove that if f is an *even* function, i.e., $f(-x) = f(x)$, $\forall x \in [-a, a]$, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx. \qquad \square$$

Exercise 9.7. (a) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(x) \geq 0$, $\forall x \in [a, b]$. Prove that

$$\int_{a}^{b} f(x)dx = 0 \Longleftrightarrow f(x) = 0, \;\; \forall x \in [a, b]. \qquad \square$$

(b) Show that for any $a < b$ there exists a continuous function $u : \mathbb{R} \to \mathbb{R}$ such that $u(x) > 0$, $\forall x \in (a, b)$, and $u(x) = 0$ $\forall x \in \mathbb{R} \setminus (a, b)$.

Hint. Think of a function u whose graph looks like a roof.

(c) Suppose that $f : [0, 1] \to \mathbb{R}$ is a continuous function such that

$$\int_{0}^{1} f(x)u(x)dx = 0,$$

for any continuous function $u : [0, 1] \to \mathbb{R}$. Prove that $f(x) = 0$, $\forall x \in [0, 1]$.

Hint. Argue by contradiction. Suppose that there exists $x_0 \in [0, 1]$ such that $f(x_0) \neq 0$, say $f(x_0) > 0$. Reach a contradiction using Theorem 6.11, and the facts (a), (b) above. $\square$

Exercise 9.8. Suppose that $f_n : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of Riemann integrable functions that converges *uniformly* on $[a, b]$ to the function $f : [a, b] \to \mathbb{R}$. We set

$$d_n := \sup_{x \in [a,b]} |f(x) - f_n(x)|.$$

(a) (Compare with Exercise 6.6.) Prove that

$$\lim_{n \to \infty} d_n = 0.$$

(b) Let $X \subset [a, b]$ be a nonempty subset of $[a, b]$. Prove that, for any $n \in \mathbb{N}$, we have

$$\operatorname{osc}(f, X) \leq \operatorname{osc}(f_n, X) + 2d_n.$$

(c) Prove that, for any partition $\boldsymbol{P}$ of $[a, b]$, and any $n \in \mathbb{N}$, we have

$$\omega(f, \boldsymbol{P}) \leq \omega(f_n, \boldsymbol{P}) + 2d_n(b - a).$$

(d) Prove that f is Riemann integrable and

$$\lim_{n \to \infty} \int_{a}^{b} f_n(x)dx = \int_{a}^{b} f(x)dx. \qquad \square$$

Exercise 9.9. (a) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous and convex function. Prove that

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

(b) Use (a) to show that for any $x > y > 0$ we have

$$\frac{1}{2y}\ln\frac{x+y}{x-y} \leq \frac{x}{x^2-y^2}.$$ □

Exercise 9.10. Consider the function $f : [0, 1] \to \mathbb{R}$, $f(x) = \frac{1}{x+1}$.

(a) Compute $\int_0^1 f(x)dx$.

(b) For $n \in \mathbb{N}$ we denote by $\boldsymbol{U}_n$ the uniform partition of order n of $[0, 1]$ and by $\underline{\xi}^{(n)}$ the sample of $\boldsymbol{U}_n$ given by

$$\underline{\xi}_k^{(n)} = \frac{k}{n}, \quad k = 1, \dots, n.$$

Describe explicitly the Riemann sum $\boldsymbol{S}(f, \boldsymbol{U}_n, \underline{\xi}^{(n)})$.

(c) Use parts (a) and (b) to compute the limit in Exercise 4.22. □

Exercise 9.11. Fix a natural number k.

(a) Prove that for any $n \in \mathbb{N}$ we have

$$1^k + 2^k + \cdots + (n-1)^k \leq \int_0^n x^k dx \leq 1^k + 2^k + \cdots + n^k.$$

(b) Use (a) to prove that

$$\lim_{n\to\infty}\frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \frac{1}{k+1}.$$ □

Exercise 9.12. Consider the function

$$F : [0, \infty) \to \mathbb{R}, \quad F(x) = \int_0^{\sqrt{x}} e^{\frac{t^2}{2}} dt.$$

Show that $F(x)$ is differentiable on $(0, \infty)$ and then compute $F'(x)$, $x > 0$. □

Exercise 9.13. Suppose $f_n : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of C^1-functions with the following properties.

(i) The sequence of derivatives $f_n' : [a, b] \to \mathbb{R}$ converges *uniformly* to a function $g : [a, b] \to \mathbb{R}$.

(ii) The sequence $f_n : [a, b] \to \mathbb{R}$ converges *pointwisely* to a function $f : [a, b] \to \mathbb{R}$.

Prove that the following hold.

(a) The sequence $f_n : [a, b] \to \mathbb{R}$ converges *uniformly* to $f : [a, b] \to \mathbb{R}$.

Hint. Define $G : [a, b] \to \mathbb{R}$, $G(x) = f(a) + \int_a^x g(t)dt$. (The function g is continuous since it is a uniform limit of continuous functions.) Since f_n' is continuous, the Fundamental Theorem of Calculus shows that

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)dt.$$

Then

$$f_n(x) - G(x) = f_n(a) - f(a) + \int_a^x \big(f_n'(t) - g(t) \big) dt.$$

Use the above equality to show that the sequence f_n converges uniformly on $[a, b]$ to G. Argue next that $G = f$.

(b) The function f is C^1 and $f' = g$, i.e., the sequence $f_n' : [a, b] \to \mathbb{R}$ converges uniformly to f'. □

Exercise 9.14. Let $L > 0$. Suppose that the power series with real coefficients

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

converges absolutely for any $|x| < L$. For every $x \in (-L, L)$ we denote by $s(x)$ the sum of the above series.

(a) Show that the function $x \mapsto s(x)$ is continuous on $(-L, L)$ and, for any $R \in (0, L)$, we have

$$\int_0^R s(x)dx = a_0 R + \frac{a_1}{2} R^2 + \frac{a_2}{3} R^3 + \cdots .$$

Hint. Use the Exercises 6.8 and 9.8.

(b) Prove that the power series

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

also converges absolutely for any $|x| < L$.

(c) Prove that $s(x)$ is differentiable on $(-L, L)$ and that

$$s'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots , \quad \forall |x| < L.$$

Hint. Use the Exercises 6.8, 9.13. □

Exercise 9.15. Consider the power series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots ,$$

and respectively,

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

(a) Prove that the above series converge absolutely for any $x \in \mathbb{R}$. Denote their sums by $a(x)$ and respectively $b(x)$.

(b) Show that the functions $a, b : \mathbb{R} \to \mathbb{R}$ are differentiable and satisfy the equalities

$$a'(x) = b(x), \quad b'(x) = -a(x).$$

Hint. Use Exercise 9.14.

(c) Show that that $a(x)$ is the unique solution of the differential equation

$$a''(x) + a(x) = 0, \quad \forall x \in \mathbb{R}$$

satisfying the condition $a(0) = 0$, $a'(0) = 1$. (Compare with Exercise 7.16.) □

Exercise 9.16. Consider the function

$$f:\mathbb{R}\to\mathbb{R},\;\; f(x)=\frac{1}{1+x^2}.$$

(a) Prove that

$$f(x)=\sum_{n=0}^\infty(-1)^n x^{2n},\;\;\forall |x|<1.$$

(b) Conclude from (a) that the Taylor series of $f(x)$ at $x_0=0$ is

$$1-x^2+x^4-x^6+\cdots.$$

Hint. Use Exercise 9.14.

(c) Deduce from (a) that

$$\arctan x=\sum_{k=0}^\infty(-1)^k\frac{x^{2k+1}}{2k+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots,\;\;\forall |x|<1. \qquad \square$$

Exercise 9.17. (a) Suppose that $f,w:[a,b]\to\mathbb{R}$ are two continuous functions satisfying the following properties.

(i) The function f is continuous.

(ii) The function w is Riemann integrable and nonnegative, i.e., $w(x)\geq 0$, $\forall x\in[a,b]$.

(iii) The integral

$$W:=\int_a^b w(x)dx$$

is strictly positive.

We set

$$m:=\inf_{x\in[a,b]}f(x),\;\; M:=\sup_{x\in[a,b]}f(x).$$

Show that

$$m\leq\frac{1}{W}\int_a^b f(x)w(x)dx\leq M,$$

and then conclude that there exists a point ξ in the *open* interval (a,b) such that

$$f(\xi)=\frac{1}{W}\int_a^b f(x)w(x)dx. \qquad \square$$

(b) Use the result in (a) to show that the Integral Remainder Formula (9.47) implies the Lagrange Remainder Formula (8.2). $\square$

Exercise 9.18. Consider the function $f:[0,2]\to\mathbb{R}$, $f(x)=1-|x-1|$, $\forall x\in[0,2]$.

(a) Sketch the graph of f.

(b) Compute $\int_0^2 f(x)dx$. $\square$

Exercise 9.19. For any natural number n we define the n-th *Legendre polynomial* to be

$$P_n(x):=\frac{1}{2^n n!}\frac{d^n}{dx^n}\big(x^2-1\big)^n.$$

We set $P_0(x)=1$, $\forall x$.

(a) Compute $P_1(x),P_2(x),P_3(x)$.

(b) Compute

$$\int_{-1}^{1} P_1(x)^2 dx, \quad \int_{-1}^{1} P_2(x)^2 dx, \quad \int_{-1}^{1} P_3(x)^2 dx, \quad \int_{-1}^{1} P_1(x) P_2(x) dx,$$

(c) Use integration-by-parts to compute

$$\int_{-1}^{1} P_m(x) P_n(x) dx, \quad \int_{-1}^{1} P_n(x)^2 dx, \quad m, n \in \mathbb{N}, \quad m \neq n.$$

Hint: You may want to use the results in Exercise 7.6 and Example 9.49. □

Exercise 9.20. Fix an integer k. Use Stirling's formula (9.59) to compute

$$\lim_{n\to\infty} \frac{\sqrt{2n}}{2^{2n}} \binom{2n}{n+k}, \quad \lim_{n\to\infty} \frac{\sqrt{2n+1}}{2^{2n+1}} \binom{2n+1}{n+k}.$$ □

Exercise 9.21. (a) For any integer $n \geq 0$ compute the numbers

$$\int_0^1 \sin^2(2\pi nt) dt \quad \int_0^1 \cos^2(2\pi nt) dt.$$

(b) Consider the function

$$f : [0,1] \to \mathbb{R}, \quad f(x) = \frac{1}{2} - \left| x - \frac{1}{2} \right|.$$

Sketch its graph and then compute

$$\int_0^1 f^2(x) dx.$$

(c) Let f be as above. For any integer $n \geq 0$ compute the numbers

$$a_n = \int_0^1 f(x) \cos(2\pi nx) dx, \quad b_n = \int_0^1 f(x) \sin(2\pi nx) dx.$$

(d) With a_n, b_n as in (c) prove that the series

$$\sum_{n\geq 1} (a_n^2 + b_n^2)$$

is convergent.[3]

Hint. When computing the above integrals it is convenient to use the change in variables $u = x - \frac{1}{2}$, some of the trig identities in Section 5.6 and Exercise 9.6. □

Exercise 9.22. Compute the area of the region depicted in Figure 9.9. □

Exercise 9.23. Prove Proposition 9.63. □

[3]A nontrivial result in the theory of Fourier series shows that

$$\int_0^1 f^2(x) dx = a_0^2 + 2 \sum_{n\geq 1} (a_n^2 + b_n^2).$$

Exercise 9.24. Consider the function

$$f:[0,2]\to\mathbb{R},\ \ f(x)=\max\{2-x,x^2\}.$$

(a) Sketch the graph of the function.

(b) Compute the area of the region between the x-axis and the graph of f.

(c) Show that the function f is piecewise C^1 and then compute the length of its graph. □

Exercise 9.25. Prove that for any $a\in(-1,0)$ and any $b>0$ the integrals

$$\int_0^1 t^a|\ln t|^b dt,\ \ \int_1^\infty t^{a-1}|\ln t|^b dt$$

are convergent. □

Exercise 9.26. Prove that the Gamma function $\Gamma:(0,\infty)\to(0,\infty)$

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$$

is continuous.

Hint. Fix $t>0$ and then use Lagrange's mean value theorem for the function $f:(0,\infty)\to\mathbb{R}$, $f(x)=t^x$. Then use Exercise 9.25 to conclude. □

9.10. Exercises for extra credit

Exercise* 9.1. Suppose that $f:[a,b]\to\mathbb{R}$ is a continuous function and $\Phi:\mathbb{R}\to\mathbb{R}$ is a convex continuous[4] function. Prove *Jensen's inequality*

$$\Phi\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\leq\frac{1}{b-a}\int_a^b\Phi\big(f(x)\big)dx. \tag{9.79}$$

□

Exercise* 9.2. Show that the improper integrals

$$\int_0^\infty\frac{\sin x}{x}dx,\ \ \int_0^\infty\sin(x^2)dx$$

are convergent. □

Exercise* 9.3. Construct a continuous function $f:[0,\infty)\to\mathbb{R}$ satisfying the following properties.

(i) $f(x)\geq 0$, $\forall x\geq 0$.

(ii) $\sup_{x\geq 0}f(x)=\infty$.

(iii) The integral $\int_0^\infty f(x)dx$ is convergent.

□

Exercise* 9.4. Suppose that $f:[0,\infty)\to\mathbb{R}$ is a C^2-function satisfying the following conditions

[4]The continuity assumption is redundant since any convex function $\mathbb{R}\to\mathbb{R}$ is automatically continuous.

(i) $f'(0) = 0$.

(ii)

$$\lim_{x\to\infty} \frac{1}{\ln x}\big(f(x) + f'(x) \big) = 0.$$

Prove that for any $\alpha \in (0,1)$ the integral $\int_0^\infty \frac{f'(x)}{x^\alpha} dx$ is convergent. $\square$

Exercise* 9.5. Suppose that $f : [1,\infty) \to \mathbb{R}$ is differentiable, the derivative $f' : [0,\infty) \to \mathbb{R}$ is increasing and

$$\lim_{x\to\infty} f'(x) = 0.$$

(For example $f(x) = \frac{1}{x}$ or $f(x) = \ln x$.) Prove that the sequence

$$S_n := \frac{1}{2} f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2} f(n) - \int_1^n f(x)dx$$

is convergent and, if S is its limit, then for any $n \in \mathbb{N}$ we have

$$\frac{f'(n)}{n} < \frac{1}{2} f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2} f(n) - \int_1^n f(x)dx - S < 0.$$ $\square$

Exercise* 9.6. Suppose that $f : [1,\infty) \to \mathbb{R}$ is differentiable, the derivative $f' : [0,\infty) \to \mathbb{R}$ is increasing and

$$\lim_{x\to\infty} f'(x) = \infty.$$

(For example, $f(x) = x^\alpha$, $\alpha > 1$.) Prove that there resists a constant $C > 0$ such for any $n \in \mathbb{N}$ we have

$$\left| \frac{1}{2} f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2} f(n) - \int_1^n f(x)dx \right| \le C|f'(n)|.$$ $\square$

Exercise* 9.7. (a) Suppose that $f : [a,b] \to [0,\infty)$ is a Riemann integrable function. For any natural numbers $k \le n$ we set

$$\delta_n := \frac{b-a}{n}, \quad f_{n,k} = f\big(a + k\delta_n\big).$$

Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n f_{n,k} = \frac{1}{b-a} \int_a^b f(x)dx,$$

$$\lim_{n\to\infty} \left(\prod_{k=1}^n f_{n,k} \right)^{\frac{1}{n}} = \exp\left(\frac{1}{b-a} \int_a^b f(x)dx \right), \quad \exp(x) := e^x.$$

(b) Fix real numbers $c, r > 0$. Denote by A_n, and respectively G_n, the arithmetic, and respectively geometric, mean of the numbers

$$c + r, c + 2r, \dots, c + nr.$$

Prove that

$$\lim_{n\to\infty} \frac{G_n}{A_n} = \frac{2}{e}.$$ $\square$

Exercise* 9.8. Prove that the sequence

$$x_n = \frac{1^n + 2^n + \cdots + n^n}{n^{n+1}}$$

is convergent and then compute its limit. $\square$

Chapter 10

Complex numbers and some of their applications

10.1. The field of complex numbers

It is well known that there exists no real number x such that $x^2 = -1$ because $x^2 \geq 0 > -1$, $\forall x \in \mathbb{R}$. Following L. Euler, we introduce an imaginary number $\boldsymbol{i}$ with the property that

$$\boldsymbol{i}^2 = -1. \tag{10.1}$$

Sometimes we write $\boldsymbol{i} = \sqrt{-1}$. The number $\boldsymbol{i}$ is called the *imaginary unit*. This bold and somewhat arbitrary move raises some troubling questions.

Can we really do this? Yes, we just did, by fiat. Where does the "number" $\boldsymbol{i}$ come from? As its name suggests, it comes from our imagination. Can't we get into some sort of trouble? This vaguely formulated question is the more serious one, but let's just admit that we won't get in any trouble. This can be argued rigorously, but requires more advanced mathematics that did not even exist during Euler's time. It took more than a century to settle this issue. During that time mathematicians found convincing semi-rigorous arguments that this construction leads to no contradictions. As Euler and its followers, we will take it on faith that this construction won't lead us to shaky grounds.

What can we do with $\boldsymbol{i}$? Following Gauss, we define the *complex numbers*. These are quantities of the form

$$z := x + y\boldsymbol{i}, \quad x, y \in \mathbb{R}.$$

The *real part* of the complex number z is

$$\mathbf{Re}\, z := x,$$

while its *imaginary part* is

$$\mathbf{Im}\, z := y.$$

The set of all the complex numbers is denoted by $\mathbb{C}$. .

The reason we are referring to the quantities $a+b\boldsymbol{i}$ as *numbers* is because we can operate with them, much like we do with real numbers. First, we can add complex numbers. If

$$z_1 := x_1 + y_1\boldsymbol{i},\ \ z_2 = x_2 + y_2\boldsymbol{i},$$

then we define

$$z_1 + z_2 = (x_1+x_2) + (y_1+y_2)\boldsymbol{i}.$$

This operation satisfies the same properties as the addition of real numbers, namely the Axioms 1-4 in Section 2.1. Note that the real numbers are special examples of complex numbers: they are the complex numbers whose imaginary part is zero.

We can also multiply complex numbers in a natural way, taking (10.1) into account. Thus

$$(x_1+y_1\boldsymbol{i})(x_2+y_2\boldsymbol{i}) = x_1x_2 + x_1y_2\boldsymbol{i} + y_1x_2\boldsymbol{i} + y_1y_2\boldsymbol{i}^2$$

$$= (x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2)\boldsymbol{i}.$$

One can check that this multiplication is commutative, associative, and distributive with respect to the addition. Moreover, the real number 1 acts as a multiplicative unit for this operation as well, and every nonzero real number z has an inverse. The construction of the inverse requires a bit of ingenuity.

To a complex number $z = x + y\boldsymbol{i}$ we associate its conjugate

$$\bar{z} = x - y\boldsymbol{i}.$$

Observe that

$$z\bar{z} = (x+y\boldsymbol{i})(x-y\boldsymbol{i}) = x^2 - (y\boldsymbol{i})^2 = x^2 + y^2.$$

The quantity $\sqrt{x^2+y^2}$ is called the *norm* of the complex number z and it is denoted by $|z|$,

$$|z| := \sqrt{x^2+y^2}.$$

Thus

$$\bar{z}z = z\bar{z} = |z|^2.$$

In particular, if $z \neq 0$, then $|z| \neq 0$ and we have

$$\frac{1}{|z|^2}\bar{z}\cdot z = z\cdot\frac{1}{|z|^2}\bar{z} = 1.$$

Thus

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}. \tag{10.2}$$

The operation of conjugation interacts well with the operations of addition and multiplication introduced above. More precisely,

$$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2,\ \ \overline{z_1z_2} = \bar{z}_1\bar{z}_2,\ \ \forall z_1, z_2 \in \mathbb{C}. \tag{10.3}$$

Moreover

$$|z_1z_2| = |z_1|\cdot|z_2|,\ \ \forall z_1, z_2 \in \mathbb{C}. \tag{10.4}$$

The simple proofs of these equalities are left to you as an exercise.

10.1.1. The geometric interpretation of complex numbers. The complex numbers have a very useful geometric interpretation. More precisely, we identify the complex number $z = x + y\boldsymbol{i}$ with the point $Z = (x, y)$ in the Cartesian plane $\mathbb{R}^2$. In turn we can identify the point Z with its position vector $\overrightarrow{OZ}$. For this reason we will often refer to $\mathbb{C}$ as the *complex plane*.

Given two complex numbers $z_1 = x_1 + y_1\boldsymbol{i}$, $z_2 = x_2 + y_2\boldsymbol{i}$ represented in the plane by the position vectors $\overrightarrow{OZ_1}$ and $\overrightarrow{OZ_2}$, then their sum $z_3 = (x_1 + x_2) + (y_1 + y_2)\boldsymbol{i}$ is represented in the plane by the point Z_3 with position vector

$$\overrightarrow{OZ_3} = \overrightarrow{OZ_1} + \overrightarrow{OZ_2},$$

where the addition of vectors is performed via the parallelogram rule; see Figure 10.1.

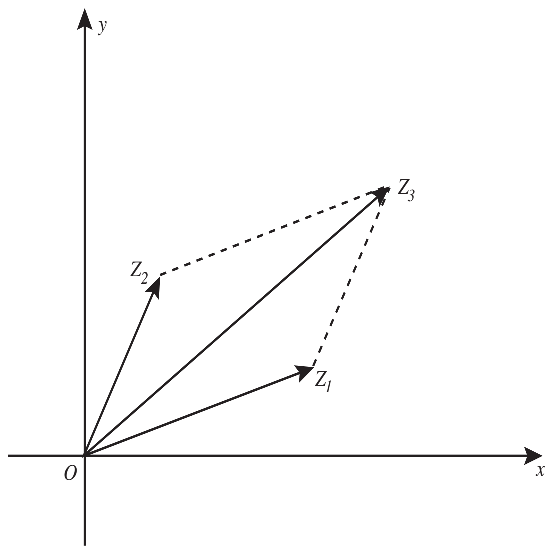

Figure 10.1. *The geometric interpretation of the sum of complex numbers.*

If the complex number $z = x + y\boldsymbol{i}$ is described by the point $Z = (x, y)$ in $\mathbb{R}^2$, then its conjugate $\bar{z} = x - y\boldsymbol{i}$ is represented by the point $Z^- = (x, -y)$, the reflection of Z in the x-axis; see Figure 10.2. Note that the norm $|z| = \sqrt{x^2 + y^2}$ is equal to the length of the vector $\overrightarrow{OZ}$,

$$|z| = \big|\overrightarrow{OZ}\big|.$$

The vector $\overrightarrow{OZ}$ makes an angle θ with the x-axis measured in a counterclockwise fashion, starting on the x-axis. Measured in radians, it can be any number in $[0, 2\pi)$. This angle is called the *argument* of the complex number z and it is denoted by $\arg z$.

Denote by r the norm of z

$$r = |z| = \sqrt{x^2 + y^2}.$$

From Figure 10.2 we deduce that the coordinates (x, y) of Z can be expressed in terms of r and θ via the equalities

$$x = r\cos\theta, \quad y = r\sin\theta,$$

so that

$$z = r\cos\theta + r\sin\theta\boldsymbol{i} = r(\cos\theta + \boldsymbol{i}\sin\theta), \quad r = |z|, \quad \theta = \arg z. \tag{10.5}$$

The equality (10.5) is usually referred to as the *trigonometric representation* of the complex number $z = x + y\boldsymbol{i}$.

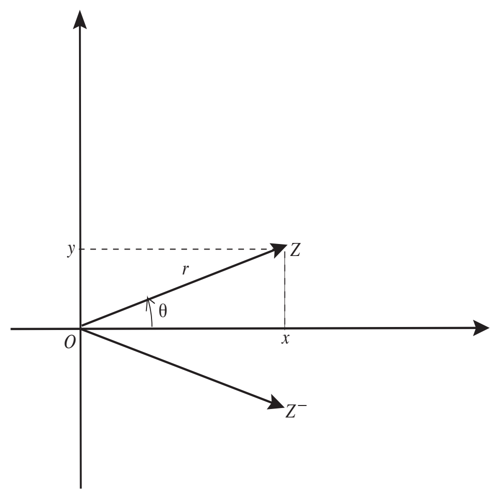

Figure 10.2. *The geometric interpretation of the conjugation of complex numbers.*

Suppose that we have two complex numbers z_1, z_2 with trigonometric representations

$$z_k = r_k(\cos\theta_k + \boldsymbol{i}\sin\theta_k), \quad r_k \geq 0, \quad k = 1, 2.$$

Then

$$\mathbf{Re}\, z_k = r_k\cos\theta_k, \quad \mathbf{Im}\, z_k = r_k\sin\theta_k.$$

Moreover

$$z_1z_2 = (r_1r_2)(\cos\theta_1 + \boldsymbol{i}\sin\theta_1)(\cos\theta_2 + \boldsymbol{i}\sin\theta_2)$$

$$= r_1r_2\Big\{ \underbrace{\big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\big)}_{=\cos(\theta_1+\theta_2)} + \boldsymbol{i}\,\underbrace{\big(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\big)}_{=\sin(\theta_1+\theta_2)} \Big\}.$$

We have thus proved that

$$r_1(\cos\theta_1 + \boldsymbol{i}\sin\theta_1) \times r_2(\cos\theta_2 + \boldsymbol{i}\sin\theta_2) = r_1r_2\big(\cos(\theta_1+\theta_2) + \boldsymbol{i}\sin(\theta_1+\theta_2)\big). \tag{10.6}$$

Applying the above equality iteratively we obtain the celebrated *Moivre's formula*

$$\big(\cos\theta + \boldsymbol{i}\sin\theta\big)^n = \cos(n\theta) + \boldsymbol{i}\sin(n\theta), \quad \forall n \in \mathbb{N}, \quad \theta \in \mathbb{R}. \tag{10.7}$$

If we combine Moivre's formula with Newton's binomial formula we can obtain many interesting consequences. We have

$$\cos n\theta + \boldsymbol{i}\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \boldsymbol{i}^k(\cos\theta)^k(\sin\theta)^{n-k}.$$

Separating the real and imaginary parts in the right-hand-side of the above equality taking into account that

$$\boldsymbol{i}^2 = -1, \quad \boldsymbol{i}^3 = -\boldsymbol{i}, \quad \boldsymbol{i}^4 = 1,$$

we deduce

$$\cos n\theta = (\cos\theta)^n - \binom{n}{2}(\cos\theta)^{n-2}(\sin\theta)^2 + \binom{n}{4}(\cos\theta)^n(\sin\theta)^4 - \cdots \tag{10.8a}$$

$$\sin n\theta = \binom{n}{1}(\cos\theta)^{n-1}\sin\theta - \binom{n}{3}(\cos\theta)^{n-3}(\sin\theta)^3 + \cdots . \tag{10.8b}$$

For example,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta,$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta,$$

$$\cos 4\theta = \cos^4\theta - \binom{4}{2}\cos^2\theta\sin^2\theta + \sin^4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \cos^4\theta,$$

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.$$

Example 10.1. Consider the complex number

$$z = \cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+\boldsymbol{i}).$$

For any $n \in \mathbb{N}$ we have

$$z^{8n} = \cos 2n\pi + \boldsymbol{i}\sin 2n\pi = 1.$$

On the other hand we have

$$z^{8n} = \frac{1}{2^{4n}}(1+\boldsymbol{i})^{8n}$$

so that

$$2^{4n} = (1+\boldsymbol{i})^{4n} = \sum_{k=0}^{8n}\binom{8n}{k}\boldsymbol{i}^k.$$

Isolating the real and imaginary parts in the right-hand side and equating them with the real and imaginary parts in the left-hand side we deduce

$$2^{4n} = \binom{8n}{0} - \binom{8n}{2} + \binom{8n}{4} - \cdots ,$$

$$0 = \binom{8n}{1} - \binom{8n}{3} + \binom{8n}{5} - \cdots . \qquad \square$$

Example 10.2. Fix a natural number $n \geq 2$. Observe that the numbers

$$\zeta_k = \cos\Big(\frac{2\pi}{k}n\Big) + \boldsymbol{i}\sin\Big(\frac{2\pi}{n}\Big), \quad k = 0, 1, \dots, n-1$$

satisfy the equation

$$\zeta_k^n = 1, \quad \forall k.$$

Conversely, if z is a complex number such that $z^n = 1$, then we deduce

$$|z|^n = 1 \Rightarrow |z| = 1,$$

and thus there exists $\theta \in [0, 2\pi)$ such that

$$z = \cos\theta + \boldsymbol{i}\sin\theta.$$

Using Moivre's formula we deduce $\cos n\theta = 1$ and $\sin n\theta = 0$ which is possible if and only if $n\theta$ is a multiple of 2π. Thus θ can only be one of the numbers

$$\frac{2\pi k}{n}, \quad k = 0, 1, \dots, n-1.$$

In other words $z^n = 1$ if and only if z is equal to one of the numbers ζ_k. For this reason the numbers ζ_k are called *the n-th roots of unity.* $\square$

10.2. Analytic properties of complex numbers

Most of the analysis we developed for real numbers carries over to complex numbers. The next result is crucial in this endeavor.

Proposition 10.3. *(a) For any complex numbers z_1, z_2 we have*

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{10.9}$$

(b) if $z = x + y\boldsymbol{i} \in \mathbb{C}$ then

$$\frac{1}{2}(|x| + |y|) \leq |z| = \sqrt{x^2 + y^2} \leq |x| + |y|. \tag{10.10}$$

Proof. (a) Let

$$z_1 = x_1 + y_1\boldsymbol{i}, \;\; z_2 = x_2 + y_2\boldsymbol{i}.$$

Then

$$|z_1| = \sqrt{x_1^2 + y_1^2}, \;\; |z_2| = \sqrt{x_2^2 + y_2^2}.$$

The Cauchy-Schwartz inequality, Corollary 8.32, implies that

$$x_1x_2 + y_1y_2 \leq \left(\sqrt{x_1^2 + y_1^2}\right) \cdot \left(\sqrt{x_2^2 + y_2^2}\right) = |z_1| \cdot |z_2|.$$

We have

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\boldsymbol{i},$$
$$|z_1 + z_2|^2 = (x_1 + y_1)^2 + (x_2 + y_2)^2$$
$$= x_1^2 + y_1^2 + 2x_1y_1 + x_2^2 + y_2^2 + 2x_2y_2 = |z_1|^2 + |z_2|^2 + 2(x_1y_1 + 2x_2y_2)$$
$$\leq |z_1|^2 + |z_2|^2 + 2|z_1| \cdot |z_2| = (|z_1| + |z_2|)^2.$$

This proves (10.9).

(b) Observe that

$$(|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x| \cdot |y| \geq |x|^2 + |y|^2 = x^2 + y^2.$$

This shows that

$$|x| + |y| \geq \sqrt{x^2 + y^2}.$$

On the other hand,

$$0 \leq (|x| - |y|)^2 = |x|^2 + |y|^2 - 2|xy| \Rightarrow 2|xy| \leq x^2 + y^2$$
$$\Rightarrow (|x| + |y|)^2 = |x|^2 + |y|^2 + 2|x| \cdot |y| \leq 2(x^2 + y^2) \Rightarrow \frac{1}{\sqrt{2}}(|x| + |y|) \leq \sqrt{x^2 + y^2}.$$

This proves (10.10). $\square$

Definition 10.4. We define the distance between two complex numbers z_1, z_2 to be the nonnegative real number

$$\operatorname{dist}(z_1, z_2) := |z_1 - z_2|. \qquad \square$$

Corollary 10.5 (The triangle inequality)**.** *For any $z_1, z_2, z_3 \in \mathbb{C}$ we have*

$$\operatorname{dist}(z_1, z_3) \leq \operatorname{dist}(z_1, z_2) + \operatorname{dist}(z_2, z_3).$$

Proof. We have

$$\begin{aligned}\operatorname{dist}(z_1, z_3) &= |z_1 - z_3| = |(z_1 - z_2) + (z_2 - z_3)| \\ &\overset{(10.9)}{\leq} |z_1 - z_2| + |z_2 - z_3| = \operatorname{dist}(z_1, z_2) + \operatorname{dist}(z_2, z_3).\end{aligned}$$

□

Definition 10.6. (a) Let $z_0 \in \mathbb{C}$ and $r > 0$. The *open disk* of center z_0 and radius r is the et

$$D_r(z_0) := \big\{ z \in \mathbb{C};\ \ \operatorname{dist}(z, z_0) < r \big\}.$$

(b) A subset $\mathcal{O} \subset \mathbb{C}$ is called *open* if for any $z_0 \in \mathcal{O}$ there exists $\varepsilon > 0$ such that

$$D_\varepsilon(z_0) \subset \mathcal{O}.$$ □

(c) A set $X \subset \mathbb{C}$ is called *closed* if the complement $\mathbb{C} \setminus X$ is open.

(d) A set $X \subset \mathbb{C}$ is called *bounded* if there exists $R > 0$ such that

$$X \subset D_R(0) \iff |z| < R,\ \ \forall z \in X.$$ □

Definition 10.7. (a) We say that a sequence of complex numbers $(z_n)_{n\geq 1}$ is *bounded* if the sequence of norms $(|z_n|)_{n\geq 1}$ is bounded as a sequence of real numbers.

(b) We say that a sequence of complex numbers $(z_n)_{n\geq 1}$ *converges to the complex number* z_*, and we denote this

$$\lim_n z_n = z_*,$$

if the sequence of nonnegative real numbers $\operatorname{dist}(z_n, z_*)$ converges to 0, i.e.,

$$\forall \varepsilon > 0\ \ \exists N = N(\varepsilon) > 0\ \ \text{such that}\ \ \forall n\ \ (n > N(\varepsilon) \Rightarrow |z_n - z_*| < \varepsilon).$$ □

Proposition 10.8. *Suppose that $(z_n)_{n\geq 1}$ is a sequence of complex numbers. We set $x_n = \mathbf{Re}\, z_n$, $y_n = \mathbf{Im}\, z_n$. The following statements are equivalent.*

(i) *The sequence (z_n) converges to the complex number $z_* = x_* + y_* \boldsymbol{i}$.*

(ii) *The sequences of* <u>real</u> *numbers $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ converge to x_* and respectively y_*.*

Proof. (i) ⇒ (ii). From the first part of (10.10) we deduce that

$$\frac{1}{2}\left(|x_n - x_*| + |y_n - y_*|\right) \leq |z_n - z_*|.$$

Since $\lim_n z_n = z_*$ we deduce $\lim_n |z_n - z_*| = 0$ and the Squeezing Principle implies

$$\lim_n \left(|x_n - x_*| + |y_n - y_*|\right) = 0.$$

The last equality implies (ii).

(ii) $\Rightarrow$ (i). From the second part of (10.10) we deduce that

$$|z_n - z_*| \leq |x_n - x_*| + |y_n - y_*|.$$

The assumption (ii) implies that

$$\lim_n \big(|x_n - x_*| + |y_n - y_*| \big) = 0.$$

From this we conclude that $\lim_n |z_n - z_*| = 0$, which is the statement (i). $\square$

Corollary 10.9. *If the sequence of complex numbers $(z_n)_{n\geq 1}$ converges to z, then*

$$\lim_n |z_n| = |z|.$$

Proof. Let $x_n := \mathbf{Re}\, z_n$ and $y_n := \mathbf{Im}\, z_n$, $x = \mathbf{Re}\, z$, $y := \mathbf{Im}\, z$. Then

$$\lim_n z_n = z \Rightarrow \lim_n x_n = x \ \wedge \ \lim_n y_n = y$$

$$\Rightarrow \lim_n (x_n^2 + y_n^2) = x^2 + y^2 \Rightarrow \lim_n \sqrt{x_n^2 + y_n^2} = \sqrt{x^2 + y^2} \iff \lim_n |z_n| = |z|.$$

$\square$

Corollary 10.10. *Any convergent sequence of complex numbers is bounded.*

Proof. Given a convergent sequence of complex numbers, the associated sequence of norms is convergent according to Corollary 10.9. The sequence of norms is thus a convergent sequence of *real* numbers, hence bounded according to Proposition 4.14. $\square$

Example 10.11. Suppose z is a complex number such that $|z| < 1$. Then

$$\lim_n z^n = 0.$$

We have to show that the sequence of nonnegative numbers $|z^n|$ goes to zero as $n \to \infty$. We set $r : |z|$ and we observe that

$$|z^n| \stackrel{(10.4)}{=} |z|^n = r^n.$$

As shown in Example 4.12

$$|r| < 1 \Rightarrow \lim_n r^n = 0 \Rightarrow \lim_n z^n = 0. \qquad \square$$

The convergent sequences of complex numbers satisfy many of the same properties of convergent sequences of real numbers. We summarize these facts in our next result whose proof is left to you as an exercise.

Proposition 10.12 (Passage to the limit)**.** *Suppose that $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two convergent sequences of complex numbers,*

$$a := \lim_{n\to\infty} a_n, \ \ b = \lim_{n\to\infty} b_n.$$

The following hold.

(i) *The sequence $(a_n + b_n)_{n\geq 1}$ is convergent and*

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b.$$

(ii) *If* $\lambda \in \mathbb{C}$ *then*

$$\lim_{n\to\infty}(\lambda a_n) = \lambda \lim_{n\to\infty} a_n = \lambda a.$$

(iii)

$$\lim_{n\to\infty}(a_n \cdot b_n) = \big(\lim_{n\to\infty} a_n\big) \cdot \big(\lim_{n\to\infty} b_n\big) = ab.$$

(iv) *Suppose that* $b \neq 0$. *Then there exists* $N_0 > 0$ *such that* $b_n \neq 0$, $\forall N > N_0$ *and*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

□

Definition 10.13. A sequence of complex numbers $(z_n)_{n\geq 1}$ is called *Cauchy* if

$$\forall \varepsilon > 0 \;\; \exists N = N(\varepsilon) > 0 \;\text{ such that }\; \forall m, n \;\; (m, n > N(\varepsilon) \Rightarrow |z_m - z_n| < \varepsilon).$$

□

The concept of Cauchy sequence of complex numbers is closely related to the notion of Cauchy sequence of real numbers. We state this in a precise form in our next result. Its proof is very similar to the proof of Proposition 10.8 and we leave the details to you as an exercise.

Proposition 10.14. *Suppose that* $(z_n)_{n\geq 1}$ *is a sequence of complex numbers. We set* $x_n := \mathbf{Re}\, z_n$, $y_n := \mathbf{Im}\, z_n$. *The following statements are equivalent.*

(i) *The sequence* $(z_n)_{n\geq 1}$ *is Cauchy.*

(ii) *The sequences of* real *numbers* $(x_n)_{n\geq 1}$ *and* $(y_n)_{n\geq 1}$ *are Cauchy.*

□

Definition 10.15. The *series* associated to a sequence $(z_n)_{n\geq 0}$ of complex numbers is the new sequence $(s_n)_{n\geq 1}$ defined by the *partial sums*

$$s_0 = z_0, \;\; s_1 = z_0 + z_1, \;\; s_2 = z_0 + z_1 + z_2, \ldots, s_n = \sum_{j=0}^{n} a_j \; \ldots .$$

The series associated to the sequence $(a_n)_{n\geq 0}$ is denoted by the symbol

$$\sum_{n\geq 0}^{\infty} z_n \quad \textit{or} \quad \sum_{n\geq 0} z_n$$

The series is called *convergent* if the sequence of partial sums $(s_n)_{n\geq 0}$ is convergent. The limit $\lim_{n\to\infty} s_n$ is called *the sum* series. We will use the notation

$$\sum_{n\geq 0} a_n = S$$

to indicate that the series is convergent and its sum is the real number S. □

Example 10.16. The geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$$

is convergent for any complex number z of norm $|z| < 1$. Indeed, its n-th partial sum is

$$s_n = 1 + z + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

If $|z| < 1$, then we deduce from Example 10.11 and Proposition 10.12 that

$$\lim_n s_n = \lim_n \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z}.$$

This shows that the series is convergent and its sum is

$$1 + z + z^2 + \cdots + z^n + \cdots = \frac{1}{1 - z}, \quad \forall |z| < 1. \tag{10.11}$$

□

Proposition 10.17. *If the series of complex numbers*

$$\sum_{n \geq 0} z_n$$

is convergent, then its terms converge to zero, $\lim_n z_n = 0$.

Proof. Denote by s the sum of the series and by s_n its n-th partial sum,

$$s_n = z_0 + z_1 + \cdots + z_n.$$

Then $z_n = s_n - s_{n-1}$ and

$$\lim_n z_n = \lim_n (s_n - s_{n-1}) = \lim_n s_n - \lim_n s_{n-1} = s - s = 0.$$

□

Example 10.18. The geometric series

$$1 + z + z^2 + \cdots$$

is divergent if $|z| \geq 1$. Indeed, we have

$$|z^n| = |z|^n$$

and

$$\lim_n |z|^n = \begin{cases} 1, & |z| = 1, \\ \infty, & |z| > 1. \end{cases}$$

This shows that when $|z| \geq 1$ the sequence (z^n) does not converge to zero and thus, according to Proposition 10.17, the geometric series cannot be convergent. □

Definition 10.19. A series of complex numbers

$$\sum_{n \geq 0} z_n$$

is called *absolutely convergent* if the series of *nonnegative real numbers*

$$\sum_{n \geq 0} |z_n|$$

is convergent. □

Proposition 10.20. *If the series of complex numbers* $\sum_{n \geq 0} z_n$ *is absolutely convergent, then it is also convergent.*

Proof. We mimic the proof of Theorem 4.46. Denote by s_n the n-th partial sum of the series $\sum_{n\geq 0} z_n$ and by $\hat{s}_n$ the n-th partial sum of the series $\sum_{n\geq 0} |z_n|$,

$$s_n = z_0 + \cdots + z_n, \ \ \hat{s}_n = |z_0| + \cdots + |z_n|.$$

For $n > m$ we have

$$s_n - s_m = z_{m+1} + \cdots + z_n, \ \ \hat{s}_n - \hat{s}_m = |z_{m+1}| + \cdots + |z_n|$$

Using (10.9) we deduce

$$|s_n - s_m| = |z_{m+1} + \cdots + z_n| \leq |z_{m+1}| + \cdots + |z_n| = \hat{s}_n - \hat{s}_m = |\hat{s}_n - \hat{s}_m|. \tag{10.12}$$

Since the series $\sum_{n\geq 0} |z_n|$ is convergent we deduce that the sequence of partial sums $(\hat{s}_n)_{n\geq 0}$ is Cauchy. Hence, for any $\varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that for any $n > m > N(\varepsilon)$ we have

$$|\hat{s}_n - \hat{s}_m| < \varepsilon.$$

Using (10.12) we deduce that for any $n > m > N(\varepsilon)$ we have

$$|s_n - s_m| < \varepsilon.$$

This shows that the sequence (s_n) is Cauchy and thus convergent according to Proposition 10.14. □

The above result reduces the problem of deciding the absolute convergence of a series of complex numbers to to deciding whether a series of nonnegative *real* numbers is convergent. We have investigated this issue in Section 4.6. We mention here one useful convergence test.

Corollary 10.21 (Ratio test)**.** *Suppose that*

$$z_0 + z_1 + z_2 + \cdots$$

is a series of complex numbers such that

$$L = \lim_n \frac{|z_{n+1}|}{|z_n|}$$

exists, $L \in [0, \infty]$*. Then the following hold.*

(i) *If* $L < 1$*, then these series* $\sum_{n\geq 0} z_n$ *is absolutely convergent.*

(ii) *If* $L > 1$*, then the series is divergent.*

Proof. (i) The ratio test Corollary 4.48 implies that the series of positive real numbers

$$\sum_{n\geq 0} |z_n|$$

is convergent.

(ii) If

$$\lim_n \frac{|z_n|}{|z_n|} > 1,$$

then $|z_{n+1}| > |z_n|$ for n sufficiently large. In particular, the sequence (z_n) does not converge to 0 and thus the series $\sum_{n\geq 0} z_n$ is divergent. □

10.3. Complex power series

A complex *power series* is a series of the form

$$s(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots = \sum_{n \geq 0} a_n z^n,$$

where z and the numbers $a_0, a_1, \dots$ are complex. The number z should be viewed as a quantity that is allowed to vary, while the numbers $a_0, a_1, \dots$ should be viewed as fixed quantities. As such they are called the *coefficients* of the power series. power series! coefficients Note that for different choices of z we obtain different series.

Example 10.22. Consider for example the power series

$$s(z) = 1 - 2z + 2^2 z^2 - 2^3 z^3 + \cdots .$$

The coefficients of this power series are

$$a_0 = 1, \;\; a_1 = -2, \;\; a_2 = 2^2, \dots, a_n = (-2)^n, \dots$$

Note that we can rewrite the above series as

$$s(z) = 1 + (-2z) + (-2z)^2 + (-2z)^3 + \cdots = \sum_{n \geq 0} (-2z)^n.$$

If we make the substitution $\zeta := -2z$ we can further rewrite

$$s(z) = 1 + \zeta + \zeta^2 + \cdots .$$

We know that this series is absolutely convergent for $|\zeta| > 1$ and divergent for $|\zeta| > 1$. In other words the power series $s(z)$ converges absolutely for $|z| < \frac{1}{2}$ and diverges fro $|z| < \frac{1}{2}$. Note that the set of complex numbers z such that $|z| < \frac{1}{2}$ is the open disk of center 0 and radius $\frac{1}{2}$. □

Proposition 10.23. *Consider a complex power series*

$$s(z) = \sum_{n \geq 0} a_n z^n.$$

(a) If for some $z_0 \neq 0$ the series $s(z_0)$ converges <u>absolutely</u>*, then for any $z \in \mathbb{C}$ such that $|z| \leq |z_0|$ the series $s(z)$ converges absolutely.*

(b) If for some $z_0 \neq 0$ the series $s(z_0)$ is convergent, <u>not necessarily absolutely</u>, then for any $z \in \mathbb{C}$ such that $|z| < |z_0|$, the series $s(z)$ converges <u>absolutely</u>*.*

Proof. (a) Since $|z| \leq |z_0|$ we deduce that

$$|a_n z^n| \leq |a_n z_0^n|, \;\; \forall n \geq 0.$$

The desired conclusion now follows from the comparison principle.

(b) Since $s(z_0)$ converges we deduce that

$$\lim_n a_n z_0^n = 0.$$

In particular, we deduce that the sequence $(a_n z_0^n)$ is bounded, i.e., there exists $C > 0$ such that

$$|a_n z_0^n| \leq C, \;\; \forall n \geq 0.$$

We set

$$r := \left|\frac{z}{z_0}\right| = \frac{|z|}{|z_0|} < 1.$$

We observe that

$$|a_n z^n| = |a_n z_0^n| \frac{|z|^n}{|z_0|^n} \leq C r^n.$$

Since $r < 1$ we deduce that the geometric series

$$\sum_{n \geq 0} C r^n$$

is convergent and the comparison principle implies that the series

$$\sum_{n \geq 0} |a_n z^n|$$

is also convergent. □

Consider a complex power series

$$s(z) = \sum_{n \geq 0} a_n z^n$$

We consider the set

$$\mathcal{R} = \big\{ r \geq 0;\ \ \exists z \in \mathbb{C} \text{ such that } \ |z| = r, \ \ s(z) \text{ is convergent} \big\} \subset \mathbb{R}.$$

Note that the set $\mathcal{R}$ is not empty because $0 \in \mathbb{R}$. Next observe that Proposition 10.23(b) implies that if $r_0 \in \mathcal{R}$, then $[0, r_0) \subset \mathcal{R}$. We set

$$R := \sup \mathcal{R} \in [0, \infty].$$

Proposition 10.23 shows that $s(z)$ converges absolutely for any $|z| < R$, and diverges for $|z| > R$. The number $R \in [0, \infty]$ is called the *radius of convergence* of the power series $s(z)$.

Example 10.24 (Complex exponential)**.** Consider the power series

$$E(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{n \geq 0} \frac{1}{n!} z^n.$$

This series is absolutely convergent for any $z \in \mathbb{C}$ because the series of positive numbers

$$\sum_{n \geq 0} \frac{|z|^n}{n!}$$

is convergent for any z. Thus the radius of convergence of this power series is ∞. For simplicity we will denote by $E(z)$ the sum of the series $E(z)$.

Observe that for a real number x the sum of the series $E(x)$ is e^x; see Exercise 8.7. We write this

$$E(x) = e^x, \ \ \forall x \in \mathbb{R}. \tag{10.13}$$

The properties of the exponential show that

$$E(x + y) = e^{x+y} = e^x e^y = E(x)E(y), \ \ \forall x, y \in \mathbb{R}. \tag{10.14}$$

A more general result is true, namely.

$$E(z + \zeta) = E(z)E(\zeta), \ \ \forall z, \zeta \in \mathbb{C}. \tag{10.15}$$

To prove the above equality we denote by $E_n(z)$ the n-th partial sum of the series $E(z)$,

$$E_n(z) = 1 + \frac{z}{1!} + \cdots + \frac{z^n}{n!}.$$

The equality (10.15) is equivalent to the equality

$$\lim_n \big(E_{2n}(z+\zeta) - E_{2n}(z)E_{2n}(\zeta) \big) = 0. \tag{10.16}$$

Fix a real number $M > 1$ such that

$$|z|,\ |\zeta| < M.$$

We have

$$E_{2n}(z+\zeta) = \sum_{m=0}^{2n} \frac{1}{m!}(z+\zeta)^m = \sum_{m=0}^{2n} \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} z^{m-j}\zeta^j$$

$$= \sum_{m=0}^{2n} \frac{1}{m!} \sum_{j=0}^{m} \frac{m! z^{m-j}\zeta^j}{(m-j)!j!} = \sum_{m=0}^{2n} \sum_{j=0}^{m} \frac{z^{m-j}\zeta^j}{(m-j)!j!}$$

$(k := m - j)$

$$= \sum_{m=0}^{2n} \sum_{\substack{j+k=m\\ j,k\geq 0}} \frac{z^k\zeta^j}{k!j!} = \sum_{\substack{j+k\leq 2n\\ j,k\geq 0}} \frac{z^k\zeta^j}{k!j!}.$$

Similarly we have

$$E_{2n}(z)E_{2n}(\zeta) = \left(\sum_{k=0}^{2n} \frac{z^k}{k!} \right) \left(\sum_{j=0}^{2n} \frac{\zeta^j}{j!} \right) = \sum_{0\leq j,k\leq 2n} \frac{z^k\zeta^j}{k!j!}.$$

We deduce

$$|E_{2n}(z+\zeta) - E_{2n}(z)E_{2n}(\zeta)| = \left| \sum_{\substack{j+k>2n\\ 0\leq j,k\leq 2n}} \frac{z^k\zeta^j}{k!j!} \right|$$

$$\leq \sum_{\substack{j+k>2n\\ 0\leq j,k\leq 2n}} \frac{|z|^k|\zeta|^j}{k!j!} \leq M^{4n} \sum_{\substack{j+k>2n\\ 0\leq j,k\leq 2n}} \frac{1}{k!j!} \leq \frac{M^{4n}}{n!} \sum_{\substack{j+k>2n\\ 0\leq j,k\leq 2n}} 1 \leq \frac{4n^2M^{4n}}{n!}.$$

From (4.8) we deduce that

$$\lim_n \frac{4n^2M^{4n}}{n!} \to 0.$$

Because of the equalities (10.13) and (10.15), for any $z \in \mathbb{C}$ we set

$$e^z := E(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} \cdots . \tag{10.17}$$

Suppose that in (10.17) the number z is purely imaginary, $z = \boldsymbol{i}t$, $t \in \mathbb{R}$. We deduce the celebrated *Euler's formula*

$$\begin{gathered} e^{\boldsymbol{i}t} = 1 + \frac{\boldsymbol{i}t}{1!} + \frac{\boldsymbol{i}^2t^2}{2!} + \frac{\boldsymbol{i}^3t^3}{3!} + \cdots \\ = \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \right) + \boldsymbol{i}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \right) \\ = \cos t + \boldsymbol{i}\sin t. \end{gathered} \tag{10.18}$$

If we let $t = \pi$ in the above equality we deduce

$$e^{\boldsymbol{i}\pi} = \cos\pi + \boldsymbol{i}\sin\pi = -1$$

i.e.,

$$e^{\boldsymbol{i}\pi} + 1 = 0. \tag{10.19}$$

The last very compact equality describes a deep connection between the five most important numbers in science: $0, 1, e, \pi, \boldsymbol{i}$. $\square$

10.4. Exercises

Exercise 10.1. Prove the equalities (10.3) and (10.4). □

Exercise 10.2. (a) Consider the complex numbers

$$z_1 = 4 + 5\boldsymbol{i}, \;\; z_2 = 5 + 12\boldsymbol{i}.$$

Compute

$$z_1 z_2, \;\; |z_2|, \;\; \frac{z_1}{z_2}.$$

(b) Show that if

$$z = \frac{1}{2}(1 + \sqrt{3}\boldsymbol{i}),$$

then

$$z^2 + z + 1 = \bar{z}^2 + \bar{z} + 1 = 0, \;\; z^3 = \bar{z}^3 = 1.$$

□

Exercise 10.3. (a) Prove that if $z \in \mathbb{C}$, then

$$z^5 = 1 \wedge z \neq 1 \iff z^4 + z^3 + z^2 + z + 1 = 0 \iff z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0.$$

(b) Suppose that z satisfies the above equation, $z^4 + z^3 + z^2 + z + 1 = 0$. We set

$$\zeta := z + \frac{1}{z}.$$

Prove that

$$z^2 + \frac{1}{z^2} = \zeta^2 - 2,$$

and

$$\zeta^2 + \zeta - 1 = 0. \tag{10.20}$$

(c) Find the two roots ζ_1, ζ_2 of the quadratic equation (10.20).

(d) If ζ_1, ζ_2 are as above, find all the complex numbers z such that

$$z + \frac{1}{z} = \zeta_1 \;\; \vee \;\; z + \frac{1}{z} = \zeta_2.$$

(e) Use (d) to compute $\cos(2\pi/5)$, $\sin(2\pi/5)$. □

Exercise 10.4. (a) Let $z_0 \in \mathbb{C}$ and $r > 0$. Prove that the open disc $D_r(z_0)$ is an open set in the sense of Definition 10.6(b).

(b) Prove that if $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{C}$ are open sets, then so are the sets $\mathcal{O}_1 \cap \mathcal{O}_2$, $\mathcal{O}_1 \cup \mathcal{O}_2$.

(c) Consider the set

$$S := \big\{ z \in \mathbb{C};\;\; \mathbf{Im}\, z = 0,\;\; \mathbf{Re}\, z \in [0,1] \big\}.$$

Draw a picture of S and then prove that it is a closed set in the sense of Definition 10.6(c).

□

Exercise 10.5. Let S be a a subset of the complex plane, $S \subset \mathbb{C}$. Prove that the following statements are equivalent.

(i) The set S is closed.

(ii) For any sequence $(z_n)_{n\geq 1}$ of points in S, $z_n \in S$, $\forall n$, if the sequence converges to z_*, then $z_* \in S$.

□

Exercise 10.6. Use the ideas in the proof of Proposition 10.8 to prove Proposition 10.14. □

Exercise 10.7. Prove Proposition 10.12 by imitating the proof of Proposition 4.15. □

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